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FRACTIONAL KINETICS IN A SPATIAL ECOLOGY MODEL

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Abstract. In this paper we study the effect of subordination to the solution of a model of spatial ecology in terms of the evolution density. The asymptotic behavior of the subordinated solution for different rates of spatial propagation is studied. The difference between subordinated solutions to non-linear equations with classical time derivative and solutions to non-linear equation with fractional time derivative is discussed.

1. Introduction

First of all we will describe the main concepts concerning kinetic behaviors for interacting particle systems in the continuum. Our description will be based essentially on [22].

Kinetic equations for classical gases may be derived from the BBGKY hierarchies for time dependent correlation functions which describe Hamiltonian dynamics of gases, see e.g. an excellent review by H. Spohn [30]. Making scalings in BBGKY hierarchical chains, we will arrive in the limiting kinetic hierarchies of Boltzmann or Vlasov type depending on the particular scaling we use. Both kinetic hierarchies have a common property of the chaos preservation. Using this property we obtain Boltzmann or Vlasov equation respectively as non-linear equations for the density of the considered system.

A similar approach may be also applied to Markov dynamics of interacting particle systems in the continuum as it was proposed in [11]. These dynamics may be described on the microscopic level by means of the related hierarchical evolution equations for correlation functions and proper scalings will lead to limiting mesoscopic hierarchies and corresponding kinetic equations. Again, a common point for the resulting hierarchies is the chaos preservation property that is a root of the kinetic equation for the density of the system. Note that this property means that the kinetic state evolution of the system will be given by a flow of Poisson measures provided the initial state is a Poisson measure. Of course, a rigorous realization of this scheme (that includes such steps as construction of the microscopic Markov dynamics, control of the convergence of solutions for the rescaled evolutions and an analysis of the corresponding kinetic equations) shall be done for each particular model and is, in general, quite difficult technical problem. At the present time, this program is realized for a number of Markov dynamics of continuous systems which includes certain birth-and-death processes, Kawasaki type dynamics, binary jumps models, see e.g. [11–13].

In the present paper we extend the approach described above to the case of certain non-Markov dynamics of interacting particle systems in the continuum. Namely, we will consider hierarchical evolution equations for correlation functions with the Caputo-Djrbashian fractional time derivatives. From the stochastic point of view, the latter corresponds to a random time change in the original Markov processes and effectively leads to a memory effect in the stochastic dynamics. The Vlasov type mesoscopic scaling

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for the fractional hierarchical chains will affect only spatial structure of their generators and will give the kinetic hierarchies of the same form as before but with fractional time derivatives. In terms of the corresponding state evolutions we obtain subordinations of Poisson flows.

The latter means that in the fractional case the kinetic hierarchies are not reduced just to density evolutions. Time development of correlation functions in such hierarchical chains is essentially different for all levels of the hierarchy. In other words, the kinetic description of the dynamics needs to work with all the hierarchy but not only with the evolution of the density. See also [9] for an overview of the fractional time kinetics.

We consider in more details the Bolker–Pacala model [6]. As mentioned above, in the Vlasov type scaling limit the first correlation function with *the classical time derivative* leads to the limiting density ρ_t, which satisfies the nonlinear evolution equation (9). While, the first correlation function with *the fractional time derivative* leads to the limiting density ρ_t^α, which corresponds to the subordination of ρ_t (see (11)). Although ρ_t solves the nonlinear equation with the classical time derivative, we do not know whether the subordinated density ρ_t^α satisfies an equation with the fractional time derivative. For instance, in general ρ_t^α will not satisfy (9) with the time derivative substituted by the fractional one (see Proposition 11 in Appendix). Such effect is different to linear equations, where subordination of a solution of a linear equation with the classical derivative solves the same linear equation with the fractional derivative (see [1]).

We want to point out that our result is an alternative to the common approach in nonlinear PDEs, when the classical time derivative is substituted by the fractional one (see e.g. [35]).

The paper is organized as follow: In Section 2 we give a brief exposition of interacting particle systems and the related fractional kinetic. Section 3 deals with the Bolker–Pacala model. In Section 4 we demonstrate that a density propagates slower after subordination. Subsection 4.2 presents particular examples of propagation rates for the density in the Bolker–Pacala model.

2. Statistical dynamics and fractional kinetic

We will consider Markov dynamics of interacting particle systems in $\mathbb{R}^d$. The phase space of such systems is the configuration space over the space $\mathbb{R}^d$ which consists of all locally finite subsets (configurations) of $\mathbb{R}^d$, namely,

$$\Gamma = \Gamma(\mathbb{R}^d) := \big\{\gamma \subset \mathbb{R}^d \big| |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)\big\}, \tag{1}$$

where $\mathcal{B}_\mathrm{b}(\mathbb{R}^d)$ denotes the family of bounded Borel subsets from $\mathbb{R}^d$. The space Γ is equipped with the vague topology, i.e., the minimal topology for which all mappings $\Gamma \ni \gamma \mapsto \sum_{x\in\gamma} f(x) \in \mathbb{R}$ are continuous for any continuous function f on $\mathbb{R}^d$ with compact support. Note that the summation in $\sum_{x\in\gamma} f(x)$ is taken over only finitely many points of γ belonging to the support of f. It was shown in [21] that with the vague topology Γ may be metrizable and it becomes a Polish space (i.e., a complete separable metric space). Corresponding to this topology, the Borel σ-algebra $\mathcal{B}(\Gamma)$ is the smallest σ-algebra for which all mappings

$$\Gamma \ni \gamma \mapsto |\gamma_\Lambda| \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$$

are measurable for any $\Lambda \in \mathcal{B}_\mathrm{b}(\mathbb{R}^d)$. Here $\gamma_\Lambda := \gamma \cap \Lambda$, and $|\cdot|$ the cardinality of a finite set. Together with Γ is useful to introduce a space Γ_0 which consists of all finite configurations in $\mathbb{R}^d$ [23].

A description of each particular model includes a heuristic Markov generator L defined on functions over the configuration space Γ of the system. We assume that the initial distribution (the state of particles) in our system is a probability measure $\mu_0 \in \mathcal{M}^1(\Gamma)$ with the corresponding sequence of correlation functions $\varkappa_0 = (k_0^{(n)})_{n=0}^\infty$, see e.g. [23] . The distribution of particles at time $t > 0$ is the measure $\mu_t \in \mathcal{M}^1(\Gamma)$, and $k_t = (k_t^{(n)})_{n=0}^\infty$

its correlation functions. If the evolution of states $(\mu_t)_{t\geq 0}$ is determined by a heuristic Markov generator L, then μ_t is the solution of the forward Kolmogorov equation (or Fokker-Plank equation (FPE)),

$$\begin{cases} \frac{\partial \mu_t}{\partial t} &= L^* \mu_t, \\ \mu_t|_{t=0} &= \mu_0, \end{cases} \tag{2}$$

where L^* is the adjoint operator. In terms of the time-dependent correlation functions $(k_t)_{t\geq 0}$ corresponding to $(\mu_t)_{t\geq 0}$, the FPE may be rewritten as an infinite system of evolution equations

$$\begin{cases} \frac{\partial k_t^{(n)}}{\partial t} &= (L^{\triangle} k_t)^{(n)}, \\ k_t^{(n)}|_{t=0} &= k_0^{(n)}, \qquad n \geq 0, \end{cases} \tag{3}$$

where $L^{\triangle}$ is the image of L^* in a space of vector-functions $k_t = (k_t^{(n)})_{n=0}^{\infty}$. In applications to concrete models, the expression for the operator $L^{\triangle}$ is obtained from the operator L via combinatorial calculations (cf. [23]).

The evolution equation (3) is nothing but a hierarchical system of equations corresponding to the Markov generator L. This system is the analogue of the BBGKY-hierarchy of the Hamiltonian dynamics [4].

Our interest now turns to Vlasov-type scaling of stochastic dynamics for the IPS in a continuum. This scaling leads to so-called kinetic description of the considered model. In the language of theoretical physics we are dealing with a mean-field type scaling which is adopted to preserve the spatial structure. In addition, this scaling will lead to the limiting hierarchy, which possesses a chaos preservation property. In other words, if the initial distribution is Poisson (non-homogeneous) then the time evolution of states will maintain this property. We refer to [11] for a general approach, concrete examples, and additional references.

There exists a standard procedure for deriving Vlasov scaling from the generator in (3). The specific type of scaling is dictated by the model in question. The process leading from $L^{\triangle}$ to the rescaled Vlasov operator $L_V^{\triangle}$ produces a non-Markovian generator L_V since it lacks the positivity-preserving property. Therefore instead of (2) we consider the following kinetic FPE:

$$\begin{cases} \frac{\partial \mu_t}{\partial t} &= L_V^* \mu_t, \\ \mu_t|t_{=0} &= \mu_0, \end{cases} \tag{4}$$

and observe that if the initial distribution satisfies $\mu_0 = \pi_{\rho_0}$, then the solution is of the same type, i.e., $\mu_t = \pi_{\rho_t}$.

In terms of correlation functions, the kinetic FPE (4) gives rise to the following Vlasov-type hierarchical chain (Vlasov hierarchy):

$$\begin{cases} \frac{\partial k_t^{(n)}}{\partial t} &= (L_V^{\triangle} k_t)^{(n)} \\ k_t^{(n)}|_{t=0} &= k_0^{(n)}, \qquad n \geq 0. \end{cases} \tag{5}$$

Let us consider the so-called Lebesgue-Poisson exponents

$$k_0(\eta) = e_\lambda(\rho_0, \eta) = \prod_{x \in \eta} \rho_0(x), \quad \eta \in \Gamma_0,$$

as the initial condition, where $\Gamma_0 \subset \Gamma$ is a subspace of finite configurations. Such correlation functions correspond to Poisson measures π_{ρ_0} on Γ with the density ρ_0. The scaling $L_V^{\triangle}$ should be such that the dynamics $k_0 \mapsto k_t$ preserves this structure, or more precisely, k_t should be of the same type

$$k_t(\eta) = e_\lambda(\rho_t, \eta) = \prod_{x \in \eta} \rho_t(x), \quad \eta \in \Gamma_0. \tag{6}$$

Relation (6) is known as the *chaos preservation property* of the Vlasov hierarchy. It turns out that equation (6) implies, in general, a non-linear differential equation

$$\frac{\partial \rho_t(x)}{\partial t} = \vartheta(\rho_t)(x), \quad x \in \mathbb{R}^d, \tag{7}$$

for ρ_t, which is called the *Vlasov-type kinetic equation.*

In general, if one does not start with a Poisson measure, the solution will leave the space $\mathcal{M}^1(\Gamma)$. To have a bigger class of initial measures, we may consider the cone inside $\mathcal{M}^1(\Gamma)$ generated by convex combinations of Poisson measures, denoted by $\mathbb{P}(\Gamma)$.

Below we discuss the concept of a fractional Fokker-Plank equation and the related fractional statistical dynamics, which is still an evolution in the space of probability measures on the configuration space. The mesoscopic scaling of this evolutions leads to a fractional kinetic FPE. A subordination principle provides for the representation of the solution to this equation as a flow of measures that is a transformation of a Poisson flow for the initial kinetic FPE.

We will introduce the fractional statistical dynamics for a given Markov generator L by changing the time derivative in the FPE to the Caputo-Djrbashian fractional derivative $\mathbb{D}^\alpha$, $\alpha \in (0,1)$ see e.g. [2]. The resulting fractional Fokker-Planck dynamics (if it exists) will act in the space of states on Γ, i.e., it will preserve probability measures on Γ. The fractional Fokker-Planck equation (FFPE)

$$\begin{cases} \mathbb{D}^\alpha \mu_t^\alpha & = L^* \mu_t^\alpha, \\ \mu_t^\alpha|_{t=0} & = \mu_0^\alpha \end{cases} \tag{FFPE}$$

describes a dynamical system with memory in the space of measures on Γ. The corresponding evolution no longer has the semigroup property. However, if the solution μ_t of equation (4) exists, then the subordination principle [1] gives us a motivation to consider the following family of measures as a solution to FFPE:

$$\mu_t^\alpha = \int_0^\infty \Phi_\alpha(\tau) \mu_{t^\alpha \tau}\, d\tau, \tag{8}$$

where Φ_α is a special case of the Wright function [1,20,24–26,33]. It is easy to see that μ_t^α is the well defined flow of measures. The FFPE equation may be written in terms of time-dependent correlation functions as an infinite system of evolution equations, the so-called *hierarchical chain*

$$\begin{cases} \mathbb{D}^\alpha k_{\alpha,t}^{(n)} & = (L^\triangle k_{\alpha,t})^{(n)}, \\ k_{\alpha,t}^{(n)}|_{t=0} & = k_{\alpha,0}^{(n)}, \qquad n \geq 0. \end{cases}$$

The evolution of the correlation functions should be expected to be given by the subordination principle. More precisely, if the solution k_t of equation (5) exists, then we may consider

$$k_{\alpha,t} = \int_0^\infty \Phi_\alpha(\tau) k_{t^\alpha \tau}\, d\tau.$$

Contrary to the subordination of the measure flow, this transformation of the correlation functions dynamics needs to be justified by certain a priori information concerning the bounds on k_t. In many particular models this information may be obtained due to the construction of the statistical dynamics (as in the model considered below).

As in the case of Markov statistical dynamics addressed above, we may consider Vlasov-type scaling in the framework of the FFPE. We know that the kinetic statistical dynamics for a Poisson initial state π_{ρ_0} is given by a flow of Poisson measures

$$\mathbb{R}_+ \ni t \mapsto \mu_t = \pi_{\rho_t} \in \mathcal{M}^1(\Gamma),$$

where ρ_t is the solution to the corresponding Vlasov kinetic equation. Then the fractional kinetic dynamics of states may be obtained as the subordination of this flow. Specifically, we consider the subordinated flow

$$\mu_t^\alpha := \int_0^\infty \Phi_\alpha(\tau)\mu_{t^\alpha \tau}\, d\tau.$$

The family of measures μ_t^α is no longer a Poisson flow. We would like to analyze the properties of these subordinated flows to distinguish the effects of fractional evolution. It is reasonable to study the properties of subordinated flows from a more general point of view when the evolution of densities $\rho_t(x)$ is not necessarily related to a particular Vlasov-type kinetic equation.

3. Microscopic spatial ecological model

Let us consider a spatial ecological model a.k.a. the Bolker-Pacala one, for the introduction and detailed study of this model see [6, 11–13, 17]. Below we formulate certain results from these papers concerning the Markov dynamics and mesoscopic scaling in the Bolker-Pacala model.

The heuristic generator in this model is

$$\begin{aligned}(LF)(\gamma) = & \sum_{x\in\gamma}\Big(m + \sum_{y\in\gamma\setminus x} a^-(x-y)\Big)[F(\gamma\setminus x) - F(\gamma)] \\ & + \sum_{x\in\gamma}\int_{\mathbb{R}^d} a^+(x-y)[F(\gamma\cup y) - F(\gamma)]dy.\end{aligned}$$

Here $m > 0$ is the mortality rate, a^- and a^+ are competition and dispersion kernels resp. Assumptions concerning these kernels we will fix later.

A standard calculation leads to the description of the correlations functions dynamics

$$\begin{cases} \frac{\partial k_t}{\partial t} & = L^\triangle k_t, \\ k_t|_{t=0} & = k_0. \end{cases}$$

As a result of the mesoscopic scaling we arrive in the following chain of equations:

$$\begin{cases} \frac{\partial k_t}{\partial t} & = L_V^\triangle k_t, \\ k_t|_{t=0} & = k_0. \end{cases}$$

This evolution of correlations functions exists in a scale of Banach spaces. We know that if $k_0 = e_\lambda(\rho_0, \cdot)$, then the solution of the above equation (chaos propagation property) is given by

$$k_t = e_\lambda(\rho_t, \cdot).$$

Under certain assumptions on the kernels $a^\pm$, the density ρ_t corresponding to a spatial ecologic logistic model, see [11], [13] and references therein, satisfies the following non-linear, non-local kinetic equation, $x \in \mathbb{R}$:

$$\frac{\partial \rho_t(x)}{\partial t} = \big(a^+ * \rho_t\big)(x) - m\rho_t(x) - \rho_t(x)\big(a^- * \rho_t\big)(x), \quad \rho_t(x)|_{t=0} = \rho_0(x), \tag{9}$$

where the initial condition ρ_0 is a bounded function. See [29] for important applications of this model in various areas of science. Next step is to consider the FFPE with Caputo-Djrbashian derivative

$$\begin{cases} \mathbb{D}_t^\alpha \mu_t^\alpha & = L^\triangle \mu_t, \\ \mu_t|_{t=0} & = \mu_0. \end{cases}$$

The corresponding evolutions for correlation functions for the Vlasov scaling is

$$\begin{cases} \mathbb{D}_t^\alpha k_{t,\alpha} & = L_V^\triangle k_{t,\alpha}, \\ k_{t,\alpha}|_{t=0} & = k_{0,\alpha}, \end{cases}$$

which is a non-Markov evolution. We would like to study some properties of the evolution $k_{t,\alpha}$. The general subordination principle gives

$$k_{t,\alpha}(\eta) = \int_0^\infty \Phi_\alpha(\tau) k_{t^\alpha\tau}(\eta)\, d\tau, \quad \eta \in \Gamma_0, \tag{10}$$

which is a relation to all orders of the correlation functions. In particular, the density of "particles" is given

$$\rho_t^\alpha(x) = k_{t,\alpha}^{(1)}(x).$$

The general subordination principle (10) gives

$$\rho_t^\alpha(x) = \int_0^\infty \Phi_\alpha(\tau) \rho_{t^\alpha\tau}(x)\, d\tau. \tag{11}$$

From this representation we shall derive an effect of the fractional derivative onto the evolution of the density.

4. Properties of the subordinated density

In this section we study long-time behavior of the subordinated density (11). In Theorem 1 we demonstrate that a function propagates slower after subordination. Then we consider examples of subordinated traveling waves and solutions to (9).

4.1. **Abstract case.** For any $0 < \alpha \le 1$ and $u : \mathbb{R} \times \mathbb{R}_+ \longrightarrow [0,1]$, u^α denotes the subordination of u by the density Φ_α and is given by (11), namely,

$$u^\alpha(x,t) = \int_0^\infty \Phi_\alpha(\tau) u(x, t^\alpha\tau)\, d\tau, \quad (x,t) \in \mathbb{R} \times \mathbb{R}_+. \tag{12}$$

Roughly speaking, the following theorem states that if the level set of u is located at $\eta(t) \in \mathbb{R}$, then the level set of u^α is located at $\eta(kt^\alpha)$, where k is a constant which depends on the level set of u^α.

Theorem 1. *Let $u : \mathbb{R}_+ \times \mathbb{R}_+ \to [0,1]$ be a continuous function and $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ be monotonically increasing to infinity such that, for any $\varepsilon \in (0,1)$,*

$$\lim_{t\to\infty} \sup_{x \ge \eta(t+\varepsilon t)} u(x,t) = 0, \tag{13}$$

$$\lim_{t\to\infty} \inf_{0 \le x \le \eta(t-\varepsilon t)} u(x,t) = 1. \tag{14}$$

Then, for any $\lambda \in (0,1)$, there exists $T = T(\lambda)$ such that, for all $t \ge T$, the level set $\theta_{\lambda,t} = \{x | u^\alpha(x,t) = \lambda\}$ is non-empty and compact, and the following asymptotic behavior holds

$$\sup_{x\in\theta_{\lambda,t}} \left| \frac{\eta^{-1}(x)}{t^\alpha} - k \right| \to 0, \quad t \to \infty, \tag{15}$$

where $k = k(\lambda)$ is such that $\int_k^\infty \Phi_\alpha(\tau)\, d\tau = \lambda$.

Remark 2. Since $\theta_{\lambda,t}$ is compact, then (15) yields

$$\min\{x | x \in \theta_{\lambda,t}\} = \eta\big(kt^\alpha + o(t^\alpha)\big), \quad t \to \infty,$$
$$\max\{x | x \in \theta_{\lambda,t}\} = \eta\big(kt^\alpha + o(t^\alpha)\big), \quad t \to \infty.$$

Remark 3. Note that (15) gives more information about asymptotic behavior of u^α, then it was assumed in (13) and (14) for u. This may be explained as follows. First, notice that if $T = t^\alpha\tau$ is fixed, then $\tau = \frac{T}{t^\alpha}$ is decreasing in t. Therefore, for any fixed $x \ge 0$, the family $\{\Delta(x,t,\varepsilon)\}_{t>0}$,

$$\Delta(x,t,\varepsilon) = \{\tau | x \in \big(\eta(t^\alpha\tau - \varepsilon t^\alpha\tau), \eta(t^\alpha\tau + \varepsilon t^\alpha\tau)\big)\}$$

is also decreasing in t and the contribution in (12) of the interval of time $\Delta(x,t,\varepsilon)$, for which the value $u(x,t^\alpha\tau)$ can not be estimated, narrows as t increases.

Proof of Theorem 1. The proof is divided into three steps. In a first step we show that it suffices to consider u, which is decreasing in x on $\mathbb{R}_+$. In a second step we obtain the asymptotic in t of $u^\alpha\big(\eta(kt^\alpha),t\big)$ for a given $k\in(0,\infty)$. In a third step we use the information derived in the second step to acquire the asymptotic for $\theta_{\lambda,t}$ claimed in the theorem.

Step 1. Let us introduce the following functions:

$$\bar{u}(x,t)=\sup_{y\ge x}u(y,t),\qquad \underline{u}(x,t)=\inf_{0\le y\le x}u(y,t),\quad x\ge 0,\quad t\ge 0.$$

Obviously $\bar{u}$ and $\underline{u}$ are decreasing in x on $\mathbb{R}_+$ and $\underline{u}\le u\le\bar{u}$. Moreover

$$\begin{aligned}
&1\ge\inf_{0\le x\le\eta(t-\varepsilon t)}\bar{u}(x,t)\ge\inf_{0\le x\le\eta(t-\varepsilon t)}u(x,t)=\underline{u}\big(\eta(t-\varepsilon t),t\big)\to 1,\quad t\to\infty,\\
&0\le\sup_{x\ge\eta(t+\varepsilon t)}\underline{u}(x,t)\le\sup_{x\ge\eta(t+\varepsilon t)}u(x,t)=\bar{u}\big(\eta(t+\varepsilon t),t\big)\to 0,\quad t\to\infty.
\end{aligned}$$

Therefore, $\underline{u}$ and $\bar{u}$ also satisfy (13)–(14) and, without loss of generality, one can assume that, for any $t\ge 0$, u is decreasing in x on $\mathbb{R}_+$.

Step 2. Let $\varepsilon>0$ be given, then for any fixed $k\in(0,\infty)$, one has

$$\begin{aligned}
u^\alpha\big(\eta(kt^\alpha),t\big)&=\int_0^\infty\Phi_\alpha(\tau)u\big(\eta(kt^\alpha),t^\alpha\tau\big)\,d\tau\\
&=\int_0^{k/1+\varepsilon}\Phi_\alpha(\tau)u\left(\eta\left((1+\varepsilon)\frac{kt^\alpha}{1+\varepsilon}\right),t^\alpha\tau\right)d\tau\\
&\quad+\int_{k/1-\varepsilon}^\infty\Phi_\alpha(\tau)u\left(\eta\left((1-\varepsilon)\frac{kt^\alpha}{1-\varepsilon}\right),t^\alpha\tau\right)d\tau\\
&\quad+\int_{k/1+\varepsilon}^{k/1-\varepsilon}\Phi_\alpha(\tau)u\big(\eta(kt^\alpha),t^\alpha\tau\big)\,d\tau\\
&=:I_1+I_2+I_3.
\end{aligned}$$

We estimate each of the above integrals. Taking into account the monotonicity of u and the assumptions (13)–(14) the following estimate for I_1 holds:

$$\begin{aligned}
0\le I_1&\le\int_0^{k/1+\varepsilon}\Phi_\alpha(\tau)u\left(\eta\left((1+\varepsilon)\tau t^\alpha\right),t^\alpha\tau\right)\,d\tau\\
&\le\int_0^{k/1+\varepsilon}\Phi_\alpha(\tau)\left(\sup_{x\ge\eta((1+\varepsilon)\tau t^\alpha)}u\big(\eta(x),t^\alpha\tau\big)\right)d\tau\\
&\to 0,\quad t\to\infty.
\end{aligned}$$

In a similar way we obtain for I_2

$$\begin{aligned}
\int_{k/1-\varepsilon}^\infty\Phi_\alpha(\tau)\,d\tau\ge I_2&\ge\int_{k/1-\varepsilon}^\infty\Phi_\alpha(\tau)u\left(\eta\left((1-\varepsilon)\tau t^\alpha\right),t^\alpha\tau\right)\,d\tau\\
&\ge\int_{k/1-\varepsilon}^\infty\Phi_\alpha(\tau)\left(\inf_{0\le x\le\eta((1-\varepsilon)\tau t^\alpha)}u(\eta(x),t^\alpha\tau)\right)d\tau,
\end{aligned}$$

and by (14), it is clear that

$$I_2\to\int_{k/1-\varepsilon}^\infty\Phi_\alpha(\tau)\,d\tau,\quad t\to\infty.$$

Finally, it is easy to prove that

$$|I_3| \le \int_{k/1+\varepsilon}^{k/1-\varepsilon} \Phi_\alpha(\tau)\, d\tau.$$

Putting all together and letting $\varepsilon \to 0$, we obtain

$$\lim_{t\to\infty} u^\alpha\big(\eta(kt^\alpha), t\big) = \int_k^\infty \Phi_\alpha(\tau)\, d\tau =: \lambda. \tag{16}$$

Notice that if $0 < \tilde{k} < k < \infty$, then

$$\lim_{t\to\infty} u^\alpha\big(\eta(\tilde{k}t^\alpha), t\big) = \int_{\tilde{k}}^\infty \Phi_\alpha(\tau)\, d\lambda = \tilde{\lambda} > \lambda.$$

Step 3. We now consider the level set $\theta_{\lambda,t} = \{x | u^\alpha(x,t) = \lambda\}$, for $\lambda \in (0,1)$. By (12), for any $t \ge 0$, u^α is continuous in x on $\mathbb{R}_+$. By (16), for any $\delta > 0$, there exists $T = T(\lambda, \delta)$ such that, for all $t \ge T$, $\theta_{\lambda,t}$ is non-empty, closed and bounded, and for all $x \in \theta_{\lambda,t}$,

$$\eta\big((k-\delta)t^\alpha\big) \le x \le \eta\big((k+\delta)t^\alpha\big).$$

Due to the monotonicity of η, for all $x \in \theta_{\lambda,t}$, one has

$$k - \delta \le \frac{\eta^{-1}(x)}{t^\alpha} \le k + \delta.$$

This completes the proof. □

Remark 4. Using a reflection of the function u, namely $\tilde{u} : \mathbb{R} \times \mathbb{R}_+ \to [0,1]$, $(x,t) \mapsto \tilde{u}(x,t) := u(-x,t)$, we could obtain the propagation information on $\mathbb{R}_-$.

4.2. **Long-time behavior.** Here and in what follows without loss of generality we may assume that the nontrivial constant solution to (9) equals 1, namely

$$\frac{\|a^+\|_{L^1} - m}{\|a^-\|_{L^1}} = 1.$$

Indeed, if $\|a^+\|_{L^1} < m$, then any solution to (9) with a bounded initial condition will tend to zero, as time tends to infinity (see e.g. [14]). For the case $\|a^+\|_{L^1} = m$ we refer the reader to [31,32]. If $\|a\|_{L^1} > m$, then nontrivial long-time behavior of the solution is possible. In this case one can always normalize (9).

Now we will give concrete examples of propagating solutions to (9) and study the asymptotic behavior in time of the corresponding subordinations.

Here and subsequently ρ will denote a continuous solution to (9) with an initial condition ρ_0. The corresponding subordination of ρ, defined by (12), will be denoted by ρ^α. For any $\lambda \in (0,1)$, we will denote by $\theta_{\lambda,t}$ the level set $\{x | \rho^\alpha(x,t) = \lambda\}$. Let us introduce the following notation of the bilateral Laplace transform

$$\big(\mathfrak{L}a^+\big)(\lambda) := \int_{\mathbb{R}} a^+(x) e^{\lambda x}\, dx. \tag{17}$$

If there exists $\lambda_0 > 0$ such that $\big(\mathfrak{L}a^+\big)(\lambda_0) < \infty$ and the initial condition ρ_0 is exponentially bounded, then it is known (see [14]) that the solution to (9) propagates with a constant speed. In this case we consider the following examples.

Example 5 (Monotone traveling wave). A function $\rho : \mathbb{R}\times\mathbb{R}_+ \to [0,1]$, which is a solution to (9), is said to be a (monotone) traveling wave with a speed $c \in \mathbb{R}$ if and only if there exists a right-continuous decreasing function $\varphi : \mathbb{R} \to [0,1]$, called the *profile* for the traveling wave, such that $\varphi(-\infty) = 1$, $\varphi(\infty) = 0$ and, for all $t \ge 0$,

$$\rho(x,t) := \varphi(x - ct), \quad \text{a.a. } x \in \mathbb{R}.$$

Theorem 4.9 and Theorem 4.33 in [14] provide existence and uniqueness results for the traveling waves to the equation (9) (see also [7, 8, 28, 34] for similar equations). Namely, under additional assumptions there exists $c_* \in \mathbb{R}$, such that for all $c \geq c_*$ there exists a unique monotone traveling wave and it does not exist if $c < c_*$.

It is proved in [14, Theorem 4.23], that the following formula for c_* holds:

$$c_* = \inf_{\lambda>0} \frac{\big(\mathfrak{L}a^+\big)(\lambda) - m}{\lambda}, \tag{18}$$

where $\mathfrak{L}a^+$ is defined by (17).

By [14, Proposition 4.11, Corollary 4.12], the profile φ of the traveling wave is of the class C_b^∞, for $c \neq 0$, and it is continuous otherwise. By [14, Theorem 3.9], φ is a strictly decreasing function. Therefore, for any $\lambda \in (0, 1)$, the level set $\theta_{\lambda,t}$ consists of one point, which we also denote by $\theta_{\lambda,t}$. By Theorem 1 with $\eta(t) = ct$, the following asymptotic for $\theta_{\lambda,t}$ holds

$$\theta_{\lambda,t} = ckt^\alpha + o(t^\alpha), \quad t \to \infty; \quad \lambda := \int_k^\infty \Phi_\alpha(\tau)\, d\tau.$$

Indeed, if $c > 0$ the result is straightforward. If $c < 0$, one can apply Theorem 1 to $u(x, t) = 1 - \varphi(ct - x)$. If $c = 0$, then $\rho^\alpha(x, t) = \rho(x, t) = \varphi(x)$.

In conclusion, if $c \neq 0$, then the subordinated traveling wave does not have a constant in time profile, since, as smaller a level set is, as faster it moves. The corresponding propagation becomes sub-linear and decreasing for $\alpha \in (0, 1)$.

Example 6 (Exponential decay). Let us now assume that the initial condition ρ_0 be such that, for all $\lambda > 0$,

$$\sup_{x\geq 0} \rho_0(x) e^{\lambda x} < \infty.$$

Then, by [14, Theorem 5.4, Theorem 5.10] the corresponding solution to (9) satisfies (13) and (14) of Theorem 1, for $\eta(t) = c_* t$, where c_* is defined by (18). Therefore the subordination of ρ will have the following asymptotic:

$$\sup_{x\in\theta_{\lambda,t}} \left| \frac{x}{c_* t^\alpha} - k \right| \to 0, \quad t \to \infty, \quad \text{where} \quad \lambda := \int_k^\infty \Phi_\alpha(\tau)\, d\tau.$$

In contrast to the previous examples, if for all $\lambda > 0$, $\big(\mathfrak{L}(a^+ * \rho_0)\big)(\lambda) = \infty$, and both a^+ and ρ_0 are regular enough, then the propagation of the corresponding solution ρ to (9) will be accelerating in time (for details see [15, 16]). In this case ρ will satisfy conditions (13) and (14) with η defined as follows

$$\ln \eta(t) \sim \ln \big(a^+ * \rho_0\big)^{(-1)}(e^{-\beta t}), \quad t \to \infty. \tag{19}$$

where $\beta := \|a^+\|_{L^1} - m$ and $f^{(-1)}$ denotes inverse of a function f. Note that η may be defined up to a logarithmic equivalent: $\ln \tilde{\eta}(t) \sim \ln \eta(t)$, $t \to \infty$.

Example 7. Let $\rho_0 \in L^1(\mathbb{R})$ and $\rho_0(x) \leq Ca^+(x)$, $x \geq x_0$, for some $C, x_0 > 0$. In this case η will depend on a^+. For $\beta := \|a^+\|_{L^1} - m$, $x \geq x_0$, $p > 0$, $q > 1$, $\gamma \in (0, 1)$ the following examples hold [15, 16] (see also [5, 18]),

$$a^+(x) = x^{-q}, \qquad \eta(t) = \exp\Big(\frac{\beta t}{q}\Big);$$

$$a^+(x) = \exp\big(-p(\ln x)^q\big), \qquad \eta(t) = \exp\Big(\big(\frac{\beta t}{p}\big)^{\frac{1}{q}}\Big);$$

$$a^+(x) = \exp(-x^\gamma), \qquad \eta(t) = (\beta t)^{\frac{1}{\gamma}};$$

$$a^+(x) = \exp\big(-x(\ln x)^{-q}\big), \qquad \eta(t) = \beta t (\ln t)^q.$$

Remark 8. It is worth pointing out that if $a^+(x) = \exp(-x^\gamma)$, $\alpha \in (0,1)$, then ρ propagates as $\eta(t) = (\beta t)^{\frac{1}{\gamma}}$, so it accelerates. On the other side ρ^α propagates as $\eta(kt^\alpha) = \sqrt[\gamma]{k\beta} t^{\frac{\alpha}{\gamma}}$. In particular, if $\alpha < \gamma$, then the propagation of ρ^α is sub-linear, if $\alpha = \gamma$, it is linear, and for $\alpha > \gamma$, it is super-linear.

Example 9. Let $\rho_0 \in L^1(\mathbb{R})$ and $a^+(x) \leq C\rho_0(x)$, $x \geq x_0$. Then the previous examples hold with a^+ substituted by ρ_0.

Remark 10. We could consider ρ_0 decreasing on $\mathbb{R}$ (instead of $\rho_0 \in L^1(\mathbb{R})$). Then, Examples 7 and 9 hold with $a^+(x)$ substituted by $\int_x^\infty a^+(y)dy$. The coefficients p and q would be changed in the first two examples of η in this case. We refer the reader to [15, 16] for details.

Appendix A. A remark on the logistic equation

Let E_α be the Mittag-Leffler function. In particular there exists a probability density Φ_α on $\mathbb{R}_+$ such that the Mittag-Leffler function is the Laplace transform of Φ_α (see [27]), namely

$$E_\alpha(-z) = \int_0^\infty \Phi_\alpha(\tau) e^{-z\tau} d\tau. \tag{20}$$

If $\rho_0 \equiv const$, then for all $t \geq 0$ the corresponding solution $\rho(\cdot, t)$ to (9) is also constant is space (see [14, Corollary 2.4]). Hence for $\|a^+\|_{L^1} - m = 1$, $\|a^-\|_{L^1} = 1$, the function $u(t) \equiv -\rho(\cdot, t)$ satisfies the following logistic ODE:

$$\frac{\partial u}{\partial t} = -u + u^2, \tag{21}$$

where $u(0) = u_0 > 0$. The subordinated function $u^\alpha(t)$ is defined by (12). The following proposition holds.

Proposition 11. *Let $0 < u_0 < \frac{1}{2}$, and u be the corresponding solution to* (21)*. Then the following asymptotics hold, as $t \to \infty$,*

$$u^\alpha(t) \sim \frac{\kappa_1}{\Gamma(1-\alpha)t^\alpha}, \qquad \mathbb{D}_t^\alpha u^\alpha(t) \sim -\frac{\kappa_2}{\Gamma(1-\alpha)t^\alpha},$$

where $B = \frac{u_0}{1-u_0}$, $\kappa_1 = \ln \frac{1}{1-B}$ and $\kappa_2 = \frac{B}{1-B}$ are positive constants.

In particular,

$$\begin{aligned} \mathbb{D}_t^\alpha u^\alpha(t) + u^\alpha(t) - (u^\alpha)^2(t) &= \frac{\kappa_1 - \kappa_2}{\Gamma(1-\alpha)} t^{-\alpha} + o(t^{-\alpha}) \\ &= (\kappa_1 - \kappa_2) u^\alpha(t) + o(u^\alpha(t)), \quad t \to \infty. \end{aligned} \tag{22}$$

Proof. First we note that $0 < u_0 < \frac{1}{2}$ if and only if $0 < B < 1$.
The following explicit formula for u holds:

$$\begin{aligned} u(t) &= \frac{u_0}{u_0 + e^t(1-u_0)} = \frac{u_0 e^{-t}}{u_0 e^{-t} + (1-u_0)} \\ &= Be^{-t} \frac{1}{Be^{-t} + 1} = Be^{-t} \sum_{j \geq 0} (-1)^j B^j e^{-jt} \\ &= -\sum_{j \geq 1} (-1)^j B^j e^{-jt}, \quad t > 0. \end{aligned} \tag{23}$$

From now on we assume that $0 < B < 1$, or equivalently $0 < u_0 < \frac{1}{2}$. By (23) and (20), one has

$$u^\alpha(t) = -\int_0^\infty \Phi_\alpha(\tau) \sum_{j \geq 1} (-1)^j B^j e^{-jt^\alpha \tau} d\tau = -\sum_{j \geq 1} (-1)^j B^j E_\alpha(-jt^\alpha). \tag{24}$$

Since the Mittag-Leffler function is entire and

$$E_\alpha(-z) \sim \frac{1}{\Gamma(1-\alpha)z}, \quad z \to \infty, \tag{25}$$

then $E_\alpha(-z)$ is bounded, for $z \geq 0$. Therefore $0 < B < 1$ yields that all series from now on will be absolutely convergent, for any $t > 0$.

We note that, for any $j \geq 1$,

$$B^j E_\alpha(-jt^\alpha) - B^{j+1} E_\alpha(-(j+1)t^\alpha) \geq B^j \int_0^\infty \Phi_\alpha(\tau)(e^{-j\tau t^\alpha} - e^{-(j+1)\tau t^\alpha})\, d\tau > 0.$$

Hence, for any $n \geq 1$,

$$-\sum_{j=1}^{2n} (-1)^j B^j E_\alpha(-jt^\alpha) \leq u^\alpha(t) \leq -\sum_{j=1}^{2n+1} (-1)^j B^j E_\alpha(-jt^\alpha).$$

In particular, for any $n \geq 1$, (25) yields,

$$\begin{aligned} 0 < \; -\frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{2n} (-1)^j \frac{B^j}{j} &\leq \liminf_{t\to\infty} t^\alpha u^\alpha(t) \\ &\leq \limsup_{t\to\infty} t^\alpha u^\alpha(t) \leq -\frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{2n+1} (-1)^j \frac{B^j}{j}. \end{aligned}$$

Therefore,

$$\lim_{t\to\infty} t^\alpha u^\alpha(t) = -\frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{\infty} (-1)^j \frac{B^j}{j} = -\frac{\ln(1-B)}{\Gamma(1-\alpha)}.$$

Since $v(t) = E_\alpha(-\lambda t^\alpha)$ solves

$$\mathbb{D}_t^\alpha v(t) = -\lambda v(t), \quad t > 0; \quad v(0) = 1,$$

the following equation holds:

$$\mathbb{D}_t^\alpha u^\alpha(t) = \sum_{j\geq 1} (-1)^j j B^j E_\alpha(-jt^\alpha), \quad t > 0.$$

We note that, for $j \geq \frac{B}{1-B}$,

$$\begin{aligned} &jB^j E_\alpha(-jt^\alpha) - (j+1)B^{j+1} E_\alpha(-(j+1)t^\alpha) \\ &\quad = \big(jB^j - (j+1)B^{j+1}\big) E_\alpha(-jt^\alpha) + (j+1)B^{j+1}\big(E_\alpha(-jt^\alpha) \\ &\qquad - E_\alpha(-(j+1)t^\alpha)\big) \\ &\quad = (j+1)B^j(\frac{j}{j+1} - B) E_\alpha(-jt^\alpha) + (j+1)B^{j+1} \\ &\qquad \times \int_0^\infty \Phi_\alpha(\tau)(e^{-j\tau t^\alpha} - e^{-(j+1)\tau t^\alpha})\, d\tau > 0. \end{aligned}$$

Thus, for any $n \geq \frac{B}{2(1-B)}$,

$$\sum_{j=1}^{2n+1} (-1)^j j B^j E_\alpha(-jt^\alpha) \leq \mathbb{D}_t^\alpha u^\alpha(t) \leq \sum_{j=1}^{2n} (-1)^j j B^j E_\alpha(-jt^\alpha)$$

In particular, for any $n \geq 1$, (25) yields,

$$0 < \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{2n+1} (-1)^j B^j \leq \liminf_{t\to\infty} t^\alpha \mathbb{D}_t^\alpha u^\alpha(t)$$

$$\leq \limsup_{t\to\infty} t^\alpha \mathbb{D}_t^\alpha u^\alpha(t) \leq \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{2n} (-1)^j B^j.$$

As a result, one has

$$\lim_{t\to\infty} t^\alpha \mathbb{D}_t^\alpha u^\alpha(t) = \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{\infty} (-1)^j B^j = -\frac{B}{\Gamma(1-\alpha)(1-B)}.$$

The proof is fulfilled. □

Corollary 12. The function u^α does not satisfy (21) with the fractional time derivative.

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