FULL INDEFINITE STIELTJES MOMENT PROBLEM AND PADÉ APPROXIMANTS

VOLODYMYR DERKACH AND IVAN KOVALYOV

Dedicated to Yu. M. Arlinskii on the occasion of his 70th birthday

ABSTRACT. Full indefinite Stieltjes moment problem is studied via the step-by-step Schur algorithm. Naturally associated with indefinite Stieltjes moment problem are generalized Stieltjes continued fraction and a system of difference equations, which, in turn, lead to factorization of resolvent matrices of indefinite Stieltjes moment problem. A criterion for such a problem to be indeterminate in terms of continued fraction is found and a complete description of its solutions is given in the indeterminate case. Explicit formulas for diagonal and sub-diagonal Padé approximants for formal power series corresponding to indefinite Stieltjes moment problem and convergence results for Padé approximants are presented.

1. INTRODUCTION

The classical Stieltjes moment problem consists in the following: given a sequence of real numbers s_j ($j \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$) find a positive measure σ with a support on $\mathbb{R}_+$, such that

$$\int_{\mathbb{R}_+} t^j d\sigma(t) = s_j, \quad j \in \mathbb{Z}_+. \tag{1.1}$$

It follows easily from (1.1) that the inequalities

$$S_n := (s_{i+j})_{i,j=0}^{n-1} \geq 0 \quad \text{and} \quad S_n^+ := (s_{i+j+1})_{i,j=0}^{n-1} \geq 0, \quad n \in \mathbb{Z}_+, \tag{1.2}$$

are necessary for solvability of the moment problem (1.1). Moreover, the inequalities (1.2) are also sufficient for solvability of the moment problem (1.1), see [1, Appendix, Section A1]. Let

$$D_n := \det S_n \quad \text{and} \quad D_n^+ := \det S_n^+, \quad n \in \mathbb{Z}_+. \tag{1.3}$$

For every solution σ of the Stieltjes moment problem its Stieltjes transform

$$f(z) = \int_{\mathbb{R}_+} \frac{d\sigma(t)}{t - z}, \quad z \in \mathbb{C} \backslash \mathbb{R}_+ \tag{1.4}$$

belongs to the class $\mathbf{N}$ of functions holomorphic on $\mathbb{C} \backslash \mathbb{R}$ with nonnegative imaginary part in $\mathbb{C}_+$ and such that $f(\bar{z}) = \overline{f(z)}$ for $z \in \mathbb{C}_+$. Moreover, f belongs to the *Stieltjes class* $\mathbf{S}$ of functions $f \in \mathbf{N}$, which admit holomorphic and nonnegative continuation to $\mathbb{R}_-$. By M. G. Krein's criterion [27]

$$f \in \mathbf{S} \iff f \in \mathbf{N} \quad \text{and} \quad zf \in \mathbf{N}.$$

Notice, that by the Hamburger–Nevanlinna theorem [1, Theorem 3.2.1] the Stieltjes moment problem can be reformulated as the following interpolation problem at ∞ for the Stieltjes transform $f(z) = \int_{\mathbb{R}_+} \frac{d\sigma(t)}{t-z}$ of the measure σ

$$(1.5) \qquad f(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n}}{z^{2n+1}} + o\left(\frac{1}{z^{2n+1}}\right), \quad z\widehat{\to}\infty.$$

The notation $z\widehat{\to}\infty$ means that $z \to \infty$ nontangentially, that is inside the sector $\varepsilon < \arg z < \pi - \varepsilon$ for some $\varepsilon > 0$.

In the pioneering paper [40] by T. Stieltjes it was shown that if D_n, D_n^+ are positive for all $n \in \mathbb{Z}_+$ and

$$(1.6) \qquad l_n := \frac{D_n^2}{D_n^+ D_{n-1}^+}, \quad m_n := \frac{(D_{n-1}^+)^2}{D_n D_{n-1}}, \quad n \in \mathbb{N},$$

then the Laurent series $-\sum_{j=0}^{\infty} \frac{s_j}{z^{j+1}}$ can be uniquely developed in the continued fraction

$$(1.7) \qquad \cfrac{1}{-zm_1 + \cfrac{1}{l_1 + \ldots \cfrac{1}{-zm_n + \cfrac{1}{l_n + \ldots}}}}.$$

The moment problem (1.1) is called *determinate*, if it has unique solution, and *indeterminate* otherwise. A solution σ of (1.1) is called canonical, if the set of polynomials is dense in $L^2(\sigma)$. As was shown in [40] the moment problem (1.1) is indeterminate, if and only if

$$(1.8) \qquad M := \sum_{i=1}^{\infty} m_i < \infty \quad \text{and} \quad L := \sum_{i=1}^{\infty} l_i < \infty.$$

As was shown in [27, Appendix, 13.2°], in the indeterminate case there exists an entire matrix valued function $W(z) = (w_{i,j}(z))_{i,j=1}^2$ such that the set of all solutions of the problem (1.1) can be parametrised by the formula

$$(1.9) \qquad \int_{\mathbb{R}_+} \frac{d\sigma(t)}{t-z} = T_{W(z)}[\tau(z)] := \frac{w_{11}(z)\tau(z) + w_{12}(z)}{w_{21}(z)\tau(z) + w_{22}(z)},$$

where τ ranges over the Stieltjes class **S**. A subset of its canonical solutions was described already in [40].

Now let us remind the indefinite versions of the classes **N** and **S**, see [30].

Definition 1.1. ([30]) A function f meromorphic on $\mathbb{C}\backslash\mathbb{R}$ with the set of holomorphy $\mathfrak{h}_f$ is said to be in the generalized Nevanlinna class $\mathbf{N}_\kappa$ ($\kappa \in \mathbb{N}$), if for every set $z_j \in \mathbb{C}_+ \cap \mathfrak{h}_f$ ($z_i \neq \bar{z}_j$, $i,j = 1, \ldots, n$) the form

$$\sum_{i,j=1}^{n} \frac{f(z_i) - \overline{f(z_j)}}{z_i - \bar{z}_j} \xi_i \bar{\xi}_j$$

has at most κ and for some choice of z_j ($j = 1, \ldots, n$) exactly κ negative squares. A functions $f \in \mathbf{N}_\kappa$ is said to belong to the generalized Stieltjes class $\mathbf{N}_\kappa^+$, if $zf \in \mathbf{N}$.

Similarly, in [9, 10] the class $\mathbf{N}_\kappa^k$ ($\kappa, k \in \mathbb{N}$) was introduced as the set of functions $f \in \mathbf{N}_\kappa$, such that $zf(z)$ belongs to the class $\mathbf{N}_k$, see also [18, 20], where the class $\mathbf{N}_0^k$ was studied.

In the present paper we consider the following problems.

Full indefinite moment problem $MP_\kappa(\mathbf{s})$. Given $\kappa \in \mathbb{Z}_+$, and an infinite sequence $\mathbf{s} = \{s_j\}_{j=0}^{\infty}$ of real numbers, describe the set $\mathcal{M}_\kappa(\mathbf{s})$ of functions $f \in \mathbf{N}_\kappa$, which satisfy (1.10) for all $\ell \in \mathbb{N}$.

Full indefinite moment problem $MP_\kappa^k(\mathbf{s})$. Given $\kappa, k \in \mathbb{Z}_+$, and an infinite sequence $\mathbf{s} = \{s_j\}_{j=0}^{\infty}$ of real numbers, describe the set $\mathcal{M}_\kappa^k(\mathbf{s}) := \mathcal{M}_\kappa(\mathbf{s}) \cap \mathbf{N}_\kappa^k$.

Indefinite moment problems $MP_\kappa(\mathbf{s})$ and $MP_\kappa^+(\mathbf{s})$ were studied in [31], [32] by the methods of extension theory of Pontryagin space symmetric operators developed in [30], [31]. In particular, it was shown in [31] that the moment problem $MP_\kappa^+(\mathbf{s})$ is solvable if and only if the number $\nu_-(S_n)$ of negative eigenvalues of S_n does not exceed κ and $S_n^+ > 0$ for all $n \in \mathbb{N}$. Further applications of the operator approach to the moment problem $MP_\kappa^k(\mathbf{s})$ were given in [10]. A reproducing kernel approach to the moment problems $MP_\kappa(\mathbf{s})$ was presented in [21]. A step-by-step algorithm of solving the moment problems $MP_\kappa(\mathbf{s})$ was elaborated in [6], [7] and [2]. Applications of the Schur algorithm to degenerate moment problem in the class $\mathbf{N}_\kappa$ were given in [14].

The step-by-step algorithm reduces the moment problems $MP_\kappa(\mathbf{s})$ and $MP_\kappa^k(\mathbf{s})$ to the following truncated moment problems:

Truncated indefinite moment problem $MP_\kappa(\mathbf{s}, \ell)$. Given $\ell, \kappa \in \mathbb{Z}_+$, and a finite sequence $\mathbf{s} = \{s_j\}_{j=0}^\ell$ of real numbers, describe the set $\mathcal{M}_\kappa(\mathbf{s}, \ell)$ of functions $f \in \mathbf{N}_\kappa$, which satisfy the asymptotic expansion

$$(1.10) \qquad f(z) = -\frac{s_0}{z} - \cdots - \frac{s_\ell}{z^{\ell+1}} + o\left(\frac{1}{z^{\ell+1}}\right), \quad z = iy, \quad y\uparrow\infty.$$

Truncated indefinite moment problem $MP_\kappa^k(\mathbf{s}, \ell)$. Given $\ell, \kappa, k \in \mathbb{Z}_+$, and a sequence $\mathbf{s} = \{s_j\}_{j=0}^\ell$ of real numbers, describe the set $\mathcal{M}_\kappa^k(\mathbf{s}, \ell)$ of functions $f \in \mathbf{N}_\kappa^k$, which satisfy (1.10). A truncated moment problem is called *even* or *odd* regarding to the oddness of the number $\ell + 1$ of given moments.

Let $\mathcal{H}$ be the set of all infinite real sequences $\mathbf{s} = \{s_j\}_{j=0}^\infty$ and let $\mathcal{H}_\kappa$ be the set of sequences $\mathbf{s} \in \mathcal{H}$, such that

$$(1.11) \qquad \nu_-(S_n) = \kappa \quad \text{for all } n \text{ big enough.}$$

Denote by $\mathcal{H}_\kappa^k$ the set of real sequences $\mathbf{s} \in \mathcal{H}_\kappa$, such that $\{s_{j+1}\}_{j=0}^\infty \in \mathcal{H}_k$, i.e.

$$(1.12) \qquad \nu_-(S_n^+) = k \quad \text{for all } n \text{ big enough.}$$

A number $n_j \in \mathbb{N}$ is said to be a *normal index* of the sequence $\mathbf{s} \in \mathcal{H}$, if $\det S_{n_j} \neq 0$. The ordered set of normal indices

$$n_1 < n_2 < \cdots < n_N$$

of the sequence $\mathbf{s}$ is denoted by $\mathcal{N}(\mathbf{s})$.

As was shown in [7] for every $\mathbf{s} \in \mathcal{H}_\kappa$ there exists a sequence of real numbers $b_i \in \mathbb{R}\backslash\{0\}$, $i \in \mathbb{N}$ and real monic polynomials

$$(1.13) \qquad a_i(z) = z^{\ell_i} + a_{\ell_i-1}^{(i)} z^{\ell_i-1} + \cdots + a_1^{(i)} z + a_0^{(i)}$$

of degree $\ell_i = n_{i+1} - n_i$, $i \in \mathbb{N}$, such that the convergents of the continued fraction

$$(1.14) \qquad \cfrac{-b_0}{a_0(z) - \cfrac{b_1}{a_1(z) - \cdots - \cfrac{b_n}{a_n(z) - \ldots}}}$$

for sufficiently large n have the asymptotic expansion (1.10) for every $\ell \in \mathbb{N}$. This fact was known already to L. Kronecker [33] and then it was reinvented in [6]. The pairs (a_i, b_i) are called atoms, see [24] and the continued fraction (1.14) is called the P–fraction, [36].

Consider the three-term recurrence relation

$$(1.15) \qquad b_j y_{j-1}(z) - a_j(z) y_j(z) + y_{j+1}(z) = 0,$$

associated with the sequence of atoms $\{a_i, b_i\}$, $i \in \mathbb{N}$, and define polynomials $P_j(z)$ and $Q_j(z)$ as solutions of the system (1.15) subject to the initial conditions

$$(1.16) \qquad P_{-1}(z) \equiv 0, \quad P_0(z) \equiv 1, \quad Q_{-1}(z) \equiv -1, \quad Q_0(z) \equiv 0.$$

The polynomials P_j and Q_j are called Lanzcos polynomials of the first and second kind. Moreover, the n-th convergent of the continued fraction (1.14) takes the form (see [24, Section 8.3.7]).

$$f^{[n]}(z) = -\frac{Q_n(z)}{P_n(z)}, \quad n \in \mathbb{N}.$$

As was shown in [7] the set $\mathcal{M}_\kappa(\mathbf{s}, 2n_j - 2)$ can be described in terms of the Lanzcos polynomials of the first and second kind.

A sequence $\mathbf{s} \in \mathcal{H}_\kappa^k$ is called *regular* (see [15]), and is designated as $\mathbf{s} \in \mathcal{H}_\kappa^{k,reg}$, if

(1.17) $$D_{n_j}^+ = \det S_{n_j}^+ \neq 0 \quad \text{for all} \quad j \in \mathbb{N}.$$

In [16] it was shown that an even indefinite Stieltjes moment problems $MP_\kappa^k(\mathbf{s}, 2n_N - 1)$ for regular sequence $\mathbf{s}$ is solvable if and only if

(1.18) $$\kappa_N := n_-(S_{n_N}) \leq \kappa \quad \text{and} \quad k_N^+ := n_-(S_{n_N}^+) \leq k.$$

For this problem one step of the Schur algorithm was split in [16] into two substeps and this leads to the expansion of $f \in \mathcal{M}_\kappa^k(\mathbf{s}, 2n_N - 2)$ into a generalized Stieltjes continued fraction

(1.19) $$f(z) = \cfrac{1}{-zm_1(z) + \cfrac{1}{l_1 + \cdots + \cfrac{1}{-zm_N(z) + \cfrac{1}{l_N + f_N(z)}}}},$$

where m_j are polynomials, $l_j \in \mathbb{R} \backslash \{0\}$ and f_N is a function from the generalized Stieltjes class $\mathbf{N}_{\kappa-\kappa_N}^{k-k_N}$, such that $f_N(z) = o(1)$ as $z \widehat{\to} \infty$. If $f \in \mathcal{M}_\kappa^k(\mathbf{s})$ then f_N has an induced asymptotic expansion

(1.20) $$f_N(z) = -\frac{s_0^{(N)}}{z} - \frac{s_1^{(N)}}{z^2} - \cdots - \frac{s_{2n}^{(N)}}{z^{2n+1}} + o\left(\frac{1}{z^{2n+1}}\right), \quad z \widehat{\to} \infty,$$

i.e. f_N is a solution of an induced moment problem $MP_{\kappa-\kappa_N}^{k-k_N}(\mathbf{s}^{(N)})$ generated by the sequence $\mathbf{s}^{(N)} = (s_i^{(N)})_{i=0}^\infty$. Then $f_N \in \mathcal{M}_\kappa^k(\mathbf{s}^{(N)})$.

Generalized Stieltjes continued fractions were studied in [15]. Associated to the continued fraction (1.19) there is a system of difference equations (see [42, Section 1])

(1.21) $$\begin{cases} y_{2j-1} - y_{2j-3} = -zm_j(z)y_{2j-2}, \\ y_{2j} - y_{2j-2} = l_j y_{2j-1}, \end{cases} \quad j \in \mathbb{N}.$$

Define the generalized Stieltjes polynomials P_j^+ and Q_j^+ of the first and second kind as solutions of the system (1.21) subject to the initial conditions

(1.22) $$P_{-1}^+(z) \equiv 0, \quad P_0^+(z) \equiv 1, \quad Q_{-1}^+(z) \equiv 1, \quad Q_0^+(z) \equiv 0.$$

The formula (1.19) for the set of solutions of the truncated Stieltjes moment problem $MP_\kappa^k(\mathbf{s}, 2n_N - 1)$ can be rewritten in terms of the generalized Stieltjes polynomials P_j^+ and Q_j^+, $j = 2N - 1, 2N$.

Theorem 1.2. ([17]) *Let a sequence* $\mathbf{s} = \{s_i\}_{i=0}^\infty \in \mathcal{H}_\kappa^k$ *be regular,* $\mathcal{N}(\mathbf{s}) = \{n_j\}_{j=1}^\infty$, *and let* P_j^+ *and* Q_j^+ *be generalized Stieltjes polynomials of the first and second kind. Then*

(i) *A nondegenerate even moment problem* $MP_\kappa^k(\mathbf{s}, 2n_N - 1)$ *is solvable, if and only if* (1.18) *holds.*

(ii) $f \in \mathcal{M}_\kappa^k(\mathbf{s}, 2n_N - 1)$ *if and only if* f *admits the representation*

(1.23) $$f(z) = \frac{Q_{2N-1}^+(z)f_N(z) + Q_{2N}^+(z)}{P_{2N-1}^+(z)f_N(z) + P_{2N}^+(z)},$$

where $f_N(z)$ *satisfies the conditions*

(1.24) $$f_N \in \mathbf{N}_{\kappa-\kappa_N}^{k-k_N^+} \quad \text{and} \quad f_N(z) = o(1), \quad z \widehat{\to} \infty.$$

In what follows for every 2×2 matrix $W = (w_{ij})_{i,j=1}^2$ we associate the linear-fractional transformation

$$(1.25) \qquad T_W[\tau] := \frac{w_{11}\tau + w_{12}}{w_{21}\tau + w_{22}}.$$

Denote by $W_{2N}(z)$ the coefficient matrix of the linear fractional transform (1.23):

$$(1.26) \qquad W_{2N}(z) = \begin{pmatrix} Q_{2N-1}^+(z) & Q_{2N}^+(z) \\ P_{2N-1}^+(z) & P_{2N}^+(z) \end{pmatrix}.$$

Then the formula (1.23) can be rewritten as

$$(1.27) \qquad f(z) = T_{W_{2N}(z)}[f_N(z)].$$

The structure of the continued fraction (1.19) leads to the following factorization of the matrix valued function $W_{2N}(z)$:

$$W_{2N}(z) = M_1(z)L_1 \ldots M_N(z)L_N,$$

where the matrices $M_j(z)$ and L_j are defined by

$$(1.28) \qquad M_j(z) = \begin{pmatrix} 1 & 0 \\ -zm_j(z) & 1 \end{pmatrix} \quad \text{and} \quad L_j = \begin{pmatrix} 1 & l_j \\ 0 & 1 \end{pmatrix}, \quad j \in \mathbb{N}.$$

Continued fractions of the form (1.19) with positive and negative masses m_j were studied by Beals, Sattinger and Szmigielski [4] in connection with the theory of multi-peakon solutions of the Camassa-Holm equation. In [22] Eckhardt and Kostenko showed that inverse spectral problem for multi-peakon solutions of the Camassa-Holm equation is solvable in the class of continued fractions of the form (1.19) with polynomials $m_j(z) = d_j z + m_j$ of formal degree 1. In [23] the spectral theory of continued fractions (1.19) was treated within a classical Hamburger moment problem associated with this sequence.

The full indefinite Stieltjes moment problem $MP_\kappa^k(\mathbf{s})$ for regular sequences $\mathbf{s} \in \mathcal{H}_\kappa^{k,reg}$ was considered in [17] via the operator approach. The moment problem $MP_\kappa^k(\mathbf{s})$ was treated in [17] as a problem of extension theory for a symmetric operator generated by this generalized Jacobi matrix. In that paper we used quite advanced tools: the theory of boundary triples developed in [25, 18, 19], and the M. G. Kreĭn theory of resolvent matrices extended in [9, 12] to the case of indefinite inner spaces.

In the present paper we are going to use elementary tools in order to make the presentation available for a wider audience. The main idea is to use the factorization formula for the coefficients matrix $W_{2j}(z)$

$$(1.29) \qquad W_{2j}(z) = W_{2N}(z)W_{2(j-N)}^{(N)}(z), \quad j > N, \quad j, N \in \mathbb{N},$$

which allows to reduce the indefinite Stieltjes moment problem $MP_\kappa^k(\mathbf{s}, 2j)$ to some classical Stieltjes moment problem $MP_0^0(\mathbf{s}^{(N)}, 2(j-N))$ with the resolvent matrix $W_{2(j-N)}^{(N)}(z)$. Then all the known results for the classical Stieltjes moment problem can be translated to the indefinite Stieltjes moment problem. In particular, in Theorem 4.2 it is shown that the problem $MP_\kappa^k(\mathbf{s})$ is indeterminate if and only if

$$(1.30) \qquad M := \sum_{j=1}^\infty m_j(0) < \infty \quad \text{and} \quad L := \sum_{j=1}^\infty l_j < \infty.$$

In the classical case, polynomials $m_j(z)$ are constants and the criterion (1.30) coincides with the well known Stieltjes criterion, see [1, Theorem 0.4].

If (1.30) is in force, then the matrix valued functions $W_{2j}(z)$ converge to an entire matrix valued function $W_\infty^+(z)$ of order $1/2$ and the linear fractional transformation (1.27) generated by the matrix valued function $W_\infty(z)$ provides a description of the set $\mathcal{M}_\kappa^k(\mathbf{s})$.

In Section 5 Padé approximants for formal power series corresponding to an indefinite Stieltjes moment problem are calculated. As was shown in [8] the diagonal Padé approximants for formal power series corresponding to an indefinite Hamburger moment problem are represented as a ratio of the Lanzcos polynomials of the 2-nd and the 1-st

kind, [8]. In Theorem 5.4 we show that the sub-diagonal Padé approximants of the corresponding formal power series is a ratio of the generalized Stieltjes polynomials of the 2-nd and the 1-st kind. In Theorem 5.6 convergence of Padé approximants is derived from the classical results using the formula (1.29).

In Section 6 the results are illustrated by an example of indefinite moment problem associated with Laguerre polynomials $L_n(z, \alpha)$ in the non-classical case $\alpha < -1$.

2. Preliminaries

2.1. **Generalized Nevanlinna functions.** A function $f \in \mathbf{N}_\kappa$ is said to belong to the class $\mathbf{N}_{\kappa,-\ell}$ ($\kappa, \ell \in \mathbb{Z}_+ := \mathbb{N} \cup \{0\}$) if f admits the asymptotic expansion (1.10) for some real numbers $s_0, \ldots, s_\ell$. Let us also set

$$(2.1) \qquad \mathbf{N}_{\kappa,-\infty} := \bigcap_{n \geq 0} \mathbf{N}_{\kappa,-2n}.$$

Every real polynomial $P(z) = p_\nu z^\nu + \cdots + p_1 z + p_0$ of degree ν belongs to a class $\mathbf{N}_{\kappa_-(P)}$, where the index $\kappa_-(P)$ can be evaluated by (see [30, Lemma 3.5])

$$(2.2) \qquad \kappa_-(P) = \begin{cases} \left[\frac{\nu+1}{2}\right], & \text{if } p_\nu < 0; \text{ and } \nu \text{ is odd }, \\ \left[\frac{\nu}{2}\right], & \text{otherwise }. \end{cases}$$

Recall, that a function $f \in \mathbf{N}_\kappa$ is said to be from the generalized Stieltjes class $\mathbf{N}_\kappa^{\pm k}$, if $z^{\pm 1} f(z)$ belongs to $\mathbf{N}_k$ ($\kappa, k \in \mathbb{Z}_+$). Let us collect some properties of generalized Nevanlinna functions, see [30], [9].

Proposition 2.1. ([30]) *Let $\kappa, \kappa_1, k \in \mathbb{Z}_+$. Then the following statements hold:*

(i) $f \in \mathbf{N}_\kappa \iff -\frac{1}{f} \in \mathbf{N}_\kappa$;

(ii) $f \in \mathbf{N}_\kappa^k \iff -\frac{1}{f} \in \mathbf{N}_\kappa^{-k}$;

(iii) $f \in \mathbf{N}_\kappa^k \iff z f(z) \in \mathbf{N}_k^{-\kappa}$;

(iv) *if $f \in \mathbf{N}_\kappa$, $f_1 \in \mathbf{N}_{\kappa_1}$ then $f + f_1 \in \mathbf{N}_{\kappa'}$, where $\kappa' \leq \kappa + \kappa_1$. If, in addition, $f(iy) = o(y)$ as $y \to \infty$ and f_1 is a polynomial, then $f + f_1 \in \mathbf{N}_{\kappa+\kappa_1}$;*

(v) *if a function $f \in \mathbf{N}_\kappa$ has an asymptotic expansion (1.5) for every $n \in \mathbb{N}$, then there exists $\kappa' \leq \kappa$, such that $\{s_j\}_{j=0}^\infty \in \mathcal{H}_{\kappa'}$.*

The notions of generalized poles of non-positive type of a function $f \in \mathbf{N}_\kappa$ were introduced in [32]. The following definitions are based on [34]. A point $\alpha \in \mathbb{R}$ is called a *generalized pole* of non-positive type (GPNT) of the function $f \in \mathbf{N}_\kappa$ with multiplicity $\kappa_\alpha(f)$ if

$$(2.3) \qquad -\infty < \lim_{z \hat{\to} \alpha} (z - \alpha)^{2\kappa_\alpha+1} f(z) \leq 0, \quad 0 < \lim_{z \hat{\to} \alpha} (z - \alpha)^{2\kappa_\alpha-1} f(z) \leq \infty.$$

Similarly, the point ∞ is called a generalized pole of f of nonpositive type (GPNT) with multiplicity $\kappa_\infty(f)$ if

$$(2.4) \qquad 0 \leq \lim_{z \hat{\to} \infty} \frac{f(z)}{z^{2\kappa_\infty+1}} < \infty, \quad -\infty \leq \lim_{z \hat{\to} \infty} \frac{f(z)}{z^{2\kappa_\infty-1}} < 0.$$

The following fundamental result was proved in [32, Theorem 3.5].

Proposition 2.2. *Let $f \in \mathbf{N}_\kappa$. Then the total multiplicity of the poles of f in $\mathbb{C}_+$ and the generalized poles of negative type of f in $\mathbb{R} \cup \{\infty\}$ is equal to κ.*

2.2. **Regular sequences and step–by–step algorithm.** In the present paper we will consider so-called regular sequences $\mathbf{s}$ from $\mathcal{H}_\kappa^k$ introduced in [15].

Definition 2.3. The sequence $\mathbf{s}$ is called regular (and is denoted as $\mathbf{s} \in \mathcal{H}_\kappa^{k,reg}$), if one of the following equivalent conditions holds:

(i) $P_j(0) \neq 0$ for every $j \in \mathbb{N}$;

(ii) $D_{n_j}^+ := \det S_{n_j}^+ \neq 0$ for every $j \in \mathbb{N}$;

(iii) $D_{n_j-1}^+ \neq 0$ for every $j \in \mathbb{N}$.

The following step–by–step algorithm of solving the indefinite Stieltjes moment problem for regular sequences $\mathbf{s} \in \mathcal{H}_\kappa^{k,reg}$ was developed in [17]. For an indefinite Hamburger moment problem such an algorithm was elaborated in [6, 7].

Let $f \in \mathcal{M}_\kappa^k(\mathbf{s})$ and let n_1 be the first normal index, i.e.

$$s_0 = \cdots = s_{n_1-2} = 0, \quad s_{n_1-1} \neq 0.$$

By Proposition 2.1 (4) $-1/f \in \mathbf{N}_\kappa^{-k}$ and

$$(2.5) \qquad -\frac{1}{f(z)} = P_1(z) + o(1) \quad as \quad z \widehat{\to} \infty,$$

where $P_1(z)$ is the Lanzcos polynomial of degree: $\deg P_1 = n_1$. Since $\mathbf{s}$ is regular, then $-l_1^{-1} := P_1(0) \neq 0$ and hence (2.5) can be rewritten as

$$(2.6) \qquad -\frac{1}{f(z)} = zm_1(z) - \frac{1}{g_1(z)} = zm_1(z) - \frac{1}{l_1 + f_1(z)}.$$

Let $\kappa_1 = \kappa_-(zm_1)$, $k_1 = \kappa_-(m_1)$, $k_1^+ = k_1 + \kappa_-(zl_1)$. Then by Proposition 2.1 (2), (4)

$$zm_1 \in \mathbf{N}_{\kappa_1}^{-k_1} \Rightarrow -\frac{1}{g_1} = -\frac{1}{f} - zm_1 \in \mathbf{N}_{\kappa-\kappa_1}^{-(k-k_1)} \Rightarrow g_1 \in \mathbf{N}_{\kappa-\kappa_1}^{k-k_1} \Rightarrow f_1 \in \mathbf{N}_{\kappa-\kappa_1}^{k-k_1^+}.$$

The formula (2.6) yields the following representation of $f \in \mathcal{M}_\kappa^k(\mathbf{s})$:

$$(2.7) \qquad f(z) = \cfrac{1}{-zm_1(z) + \cfrac{1}{l_1 + f_1(z)}},$$

where $f_1 \in \mathbf{N}_{\kappa-\kappa_1}^{k-k_1^+}$ satisfies the asymptotic expansion

$$(2.8) \qquad f_1(z) = -\frac{s_0^{(1)}}{z} - \frac{s_1^{(1)}}{z^2} - \cdots - \frac{s_{2n}^{(1)}}{z^{2n+1}} + o\left(\frac{1}{z^{2n+1}}\right), \quad z \widehat{\to} \infty$$

with some real numbers $s_i^{(1)}$, $i \in \mathbb{Z}_+$. Explicit formulas for calculation of the sequence $\mathbf{s}^{(1)} = \{s_i^{(1)}\}_{i=0}^\infty$ are presented in [17, Remarks 3.4, 3.6], the polynomial $m_1(z)$ and the number l_1 can be found by (see [17, (2.16), (3.27)]):

$$(2.9) \qquad m_1(z) = \frac{(-1)^{n_1+1}}{D_{n_1}} \begin{vmatrix} 0 & \dots & 0 & s_{n_1-1} & s_{n_1} \\ \vdots & & \dots & \dots & \vdots \\ s_{n_1-1} & \dots & \dots & \dots & s_{2n_1-2} \\ 1 & z & \dots & z^{n_1-2} & z^{n_1-1} \end{vmatrix},$$

$$(2.10) \qquad l_1 = (-1)^{n_1+1} s_{n_1-1} \frac{D_{n_1}}{D_{n_1}^+}, \quad D_\nu := \det S_\nu, \quad D_\nu^+ := \det S_\nu^+.$$

This completes the first step of the Schur algorithm.

Applying this algorithm repeatedly N times one obtains a function $f_N \in \mathbf{N}_{\kappa-\kappa_N}^{k-k_N^+}$, with κ_N and k_N given by (1.18), connected with f by the formula (1.19). Moreover, the function f_N satisfies the asymptotic expansion

$$(2.11) \qquad f_N(z) = -\frac{s_0^{(N)}}{z} - \frac{s_1^{(N)}}{z^2} - \cdots - \frac{s_{2n}^{(N)}}{z^{2n+1}} + o\left(\frac{1}{z^{2n+1}}\right), \quad z \widehat{\to} \infty$$

with some real $s_i^{(N)}$, $i \in \mathbb{Z}_+$.

Lemma 2.4. ([15]) *Let* $\mathbf{s} \in \mathcal{H}_\kappa^{k,reg}$. *Then there exist sequences of polynomials* $m_j(z)$ *and numbers* l_j *such that the* $2j-th$ *convergent* $\frac{u_{2j}}{v_{2j}}$ *of the continued fraction*

$$(2.12) \qquad \cfrac{1}{-zm_1(z) + \cfrac{1}{l_1 + \ldots \cfrac{1}{-zm_j(z) + \cfrac{1}{l_j + \ldots}}}}$$

coincides with the $j-th$ *convergent of the* $P-$*fraction* (1.14) *corresponding to the sequence* $\mathbf{s}$. *Let functions* f *and* f_N ($N \in \mathbb{N}$) *be connected by* (1.19) *and let* $\mathbf{s}^{(N)}$ *is the* $N-th$ *induced sequence. Then*

$$(2.13) \qquad f \in \mathcal{M}_\kappa^k(\mathbf{s}, \ell) \Longleftrightarrow f_N \in \mathcal{M}_{\kappa - \kappa_N}^{k - k_N^+}(\mathbf{s}^{(N)}, \ell - 2n_N),$$

$$(2.14) \qquad f \in \mathcal{M}_\kappa^k(\mathbf{s}) \Longleftrightarrow f_N \in \mathcal{M}_{\kappa - \kappa_N}^{k - k_N^+}(\mathbf{s}^{(N)}),$$

where κ_N *and* k_N^+ *are given by* (1.18).

In the case of a regular sequence $\mathbf{s} \in \mathcal{H}_\kappa^k$ the parameters l_j and $m_j(z)$ in (2.12) can be calculated recursively by the above Schur algorithm in terms of the sequence $\mathbf{s}$:

$$(2.15) \qquad m_j(z) = \frac{(-1)^{\nu+1}}{D_\nu^{(j-1)}} \begin{vmatrix} 0 & \ldots & 0 & s_{\nu-1}^{(j-1)} & s_\nu^{(j-1)} \\ \vdots & & \ldots & \ldots & \vdots \\ s_{\nu-1}^{(j-1)} & \ldots & \ldots & \ldots & s_{2\nu-2}^{(j-1)} \\ 1 & z & \ldots & z^{\nu-2} & z^{\nu-1} \end{vmatrix},$$

where $D_\nu^{(j)} := \det S_\nu^{(j)}$, $\nu = n_j - n_{j-1}$ and

$$(2.16) \qquad l_j = (-1)^{\nu+1} \frac{D_\nu^{(j-1)}}{\left(D_\nu^{(j-1)}\right)^+}, \quad j = 1, \ldots, N-1.$$

The $N-$th induced sequence $\mathbf{s}^{(N)}$ can be found as the sequence of coefficients of the series expansion $-\sum_{i=0}^\infty s_i^{(N)} z^{-(i+1)}$ corresponding to the continued fraction

$$(2.17) \qquad \cfrac{1}{-zm_{N+1}(z) + \cfrac{1}{l_{N+1} + \cfrac{1}{-zm_{N+2}(z) + \cfrac{1}{\ddots}}}}.$$

2.3. **Generalized Stieltjes continued fractions.** In [15] the expansion (2.12) of $f \in \mathcal{M}_\kappa^k(\mathbf{s})$ into a generalized Stieltjes fraction for $\mathbf{s} \in \mathcal{H}_\kappa^{k,reg}$ was derived from the expansion of its unwrapping transform $zf(z^2)$ into the $P-$fraction.

Theorem 2.5. ([15]) *Let* $\mathbf{s} \in \mathcal{H}_\kappa^{k,reg}$ *and let the* $P-$*fraction* (1.14) *and the generalized* $S-$*fraction* (2.12) *correspond to the sequence* $\mathbf{s}$. *Then the parameters* l_j *and* $m_j(z)$ ($j \in \mathbb{Z}_+$) *of the generalized* $\mathbf{S}-$*fraction* (2.12) *are connected with the parameters* b_j *and* $a_j(z)$ ($j \in \mathbb{N}$) *of the* $P-$*fraction* (1.14) *by the equalities*

$$(2.18) \qquad b_0 = \frac{1}{d_1}, \quad a_0(z) = \frac{1}{d_1}\left(zm_1(z) - \frac{1}{l_1}\right),$$

$$(2.19) \qquad b_j = \frac{1}{l_j^2 d_j d_{j+1}}, \quad a_j(z) = \frac{1}{d_{j+1}}\left(zm_{j+1}(z) - \left(\frac{1}{l_j} + \frac{1}{l_{j+1}}\right)\right),$$

where d_j *is the leading coefficient of* $m_j(z)$ ($j = 1, \ldots, N-1$).

In the case when $m_j(z) \equiv m_j$ are constants the generalized $S-$fraction reduces to the classical S-fraction (1.7) and the formulas (2.18) and (2.19) coincide with the well known classical formulas from [40], see also [1, Appendix, (3), (4)].

Conversely, m_j and l_j can be represented in terms of the $P-$fractions, see [15, Corollary 4.1].

Corollary 2.6. *Let* $\mathbf{s} \in \mathcal{H}_\kappa^{k,reg}$, *let* $\mathbf{s}$ *be associated with the* $\mathbf{S}-$*fraction* (2.12), *let* d_i *be the leading coefficient of the polynomial* $m_i(z)$ *and let*

$$(2.20) \qquad \qquad \widetilde{b}_i = b_0 b_1 \ldots b_i, \quad i \in \mathbb{Z}_+.$$

Then

$$(2.21) \quad d_i = \frac{P_{i-1}^2(0)}{\widetilde{b}_{i-1}}, \quad l_i = -\frac{\widetilde{b}_{i-1}}{P_{i-1}(0)P_i(0)}, \quad m_i(z) = d_i \frac{a_{i-1}(z) - a_{i-1}(0)}{z}, \quad i \in \mathbb{N}.$$

Remark 2.7. If $\mathcal{N}(\mathbf{s}) = \mathbb{N}$, then $\deg(m_i) = 0$ for all $i \in \mathbb{N}$, and $d_i = m_i$, l_i can be found by (2.21). Alternatively, m_i, l_i can be found by (1.6).

If $D_i^+ > 0$ for all $i \in \mathbb{N}$, then $\deg(m_i) \leq 1$ for all $i \in \mathbb{N}$, and $m_i(z) = d_i z + m_i$, l_i can be found by (2.21). The parameters d_i, m_i and l_i can be also expressed in terms of D_i and D_i^+, see [32, Section 5.3], [15, Remark 4.1].

3. TRUNCATED INDEFINITE MOMENT PROBLEMS

3.1. **A system of difference equations and generalized Stieltjes polynomials.**
Let us consider a system of difference equations associated with the continued fraction (2.12)

$$(3.1) \qquad \begin{cases} y_{2j-1} - y_{2j-3} = -zm_j(z)y_{2j-2}, \\ y_{2j} - y_{2j-2} = l_j y_{2j-1}, \end{cases} \quad j \in \mathbb{N}.$$

If the j–th convergent of this continued fraction is denoted by $\frac{u_j}{v_j}$, then u_j, v_j can be found as solutions of the system (3.1) (see [42, Section 1]) subject to the following initial conditions

$$(3.2) \qquad \qquad u_{-1} \equiv 1, \quad u_0 \equiv 0; \qquad v_{-1} \equiv 0, \quad v_0 \equiv 1.$$

The first two convergents of the continued fraction (2.12) take the form

$$\frac{u_1}{v_1} = \frac{1}{-zm_1(z)} = T_{M_1}[\infty], \quad \frac{u_2}{v_2} = \frac{l_1}{-zl_1m_1(z) + 1} = T_{M_1 L_1}[0].$$

Definition 3.1. ([16]) Let $\mathbf{s} \in \mathcal{H}_\kappa^{k,reg}$. Define polynomials $P_i^+(z)$, $Q_i^+(z)$ by

$$P_{-1}^+(z) \equiv 0, \quad P_0^+(z) \equiv 1, \qquad Q_{-1}^+(z) \equiv 1, \quad Q_0^+(z) \equiv 0,$$

$$(3.3) \qquad P_{2i-1}^+(z) = -\frac{1}{\widetilde{b}_{i-1}} \begin{vmatrix} P_i(z) & P_{i-1}(z) \\ P_i(0) & P_{i-1}(0) \end{vmatrix} \quad \text{and} \quad P_{2i}^+(z) = \frac{P_i(z)}{P_i(0)},$$

$$Q_{2i-1}^+(z) = \frac{1}{\widetilde{b}_{i-1}} \begin{vmatrix} Q_i(z) & Q_{i-1}(z) \\ P_i(0) & P_{i-1}(0) \end{vmatrix} \quad \text{and} \quad Q_{2i}^+(z) = -\frac{Q_i(z)}{P_i(0)}.$$

The polynomials $P_i^+(z)$, $Q_i^+(z)$ are called *the generalized Stieltjes polynomials* corresponding to the sequence $\mathbf{s}$.

As was noticed in [16] the generalized Stieltjes polynomials coincide with the solutions u_i and v_i of the system (3.1).

Proposition 3.2. *Let $s \in \mathcal{H}_\kappa^{k,reg}$ and let $P_j^+(z)$ and $Q_j^+(z)$ be the generalized Stieltjes polynomials defined by (3.3). Then the solutions $\{u_j\}_{j=0}^N$ and $\{v_j\}_{j=0}^N$ of the system (3.1), (3.2) take the form*

$$u_j = Q_j^+(z), \quad v_j = P_j^+(z), \quad j = -1, 0, \ldots, N.$$

Remark 3.3. The Stieltjes polynomials satisfy the following properties

$$(3.4) \qquad\qquad P_{2i-1}^+(0) = 0, \quad P_{2i-2}^+(0) = 1 \quad \text{and} \quad Q_{2i-1}^+|(0) = 1.$$

Obviously, by Definition 3.1

$$(3.5) \qquad P_{2i-1}^+(0) = -\frac{1}{\widetilde{b}_{i-1}} \begin{vmatrix} P_i(0) & P_{i-1}(0) \\ P_i(0) & P_{i-1}(0) \end{vmatrix} = 0 \quad \text{and} \quad P_{2i-2}^+(0) = \frac{P_i(0)}{P_i(0)} = 1.$$

By (3.7)

$$(3.6) \qquad Q_{2i-1}^+(0) = \frac{1}{\widetilde{b}_{i-1}} \begin{vmatrix} Q_i(0) & Q_{i-1}(0) \\ P_i(0) & P_{i-1}(0) \end{vmatrix} = \frac{Q_i(0)P_{i-1}(0) - Q_{i-1}(0)P_i(0)}{\widetilde{b}_{i-1}} = 1.$$

Here we used the generalized Liouville-Ostrogradsky formula (see [17, (2.9)])

$$(3.7) \qquad\qquad Q_i(z)P_{i-1}(z) - Q_{i-1}(z)P_i(z) = \widetilde{b}_{i-1}, \quad i \in \mathbb{N}.$$

Lemma 3.4. *Let P_i^+ and Q_i^+ be the Stieltjes polynomials defined by (3.3). Then*

$$(3.8) \qquad\qquad P_{2i}^+(z)Q_{2i-1}^+(z) - Q_{2i}^+(z)P_{2i-1}^+(z) = 1.$$

Proof. By Definition 3.1 and (3.7) we obtain

$$\begin{aligned} P_{2i}^+(z)Q_{2i-1}^+(z) - Q_{2i}^+(z)P_{2i-1}^+(z) &= \frac{P_i(z)}{\widetilde{b}_{i-1}P_i(0)}(Q_i(z)P_{i-1}(0) - P_i(0)Q_{i-1}(z)) - \\ &\quad - \frac{Q_i(z)}{\widetilde{b}_{i-1}P_i(0)}(P_i(z)P_{i-1}(0) - P_i(0)P_{i-1}(z)) \\ &= \frac{Q_i(z)P_{i-1}(z) - P_i(z)Q_{i-1}(z)}{\widetilde{b}_{i-1}} = 1. \end{aligned}$$

This completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 3.5. *Let $s \in \mathcal{H}_\kappa^{k,reg}$ and let $P_i(z)$ and $Q_i(z)$ $(i \in \mathbb{Z}_+)$ be Lanczos polynomials of the first and second kind and let l_j and $m_j(z)$ $(j \in \mathbb{N})$ be parameters of the generalized $\mathbf{S}$−fraction (2.12). Then*

(i) *The constants l_i can be calculated by*

$$(3.9) \qquad\qquad l_i = -\frac{Q_i(0)}{P_i(0)} + \frac{Q_{i-1}(0)}{P_{i-1}(0)}, \quad i \in \mathbb{N};$$

(ii) *For every $N \in \mathbb{N}$ the following formulas hold*

$$(3.10) \qquad\qquad \sum_{i=1}^N l_i = -\frac{Q_N(0)}{P_N(0)}, \quad \sum_{i=1}^N d_i = \sum_{i=0}^{N-1} |P_i^2(0)|\widetilde{b}_i^{-1},$$

$$(3.11) \qquad\qquad \sum_{i=1}^N m_i(0) = -P_{2N-1}^{+\,'}(0).$$

Proof. 1) Let $Q_i^+(z)$ be Stieltjes polynomials defined by (3.3). Substituting in (3.1) $y_j = Q_j^+$ and $z = 0$, we obtain

$$Q_{2i}^+(0) - Q_{2i-2}^+(0) = l_i Q_{2i-1}^+(0).$$

By Definition 3.1 and by the generalized Liouville-Ostrogradsky formula (3.7)

$$Q_{2i-1}^+(0) = \frac{1}{\widetilde{b}_{i-1}}(Q_i(0)P_{i-1}(0) - Q_{i-1}(0)P_i(0) = 1.$$

This implies (3.9).

2) Summing the equalities (3.9) for $i = 1, \ldots, N$ one obtains the first equality in (3.10). The second equality in (3.10) is implied by the relation (see [15, Corollary 4.1])

$$(3.12) \qquad d_{i+1} = |P_i^2(0)|\widetilde{b}_i^{-1}.$$

3) Differentiating the first equality in (3.1) one obtains

$$P_{2i-1}^{+'}(z) = -m_i(z)P_{2i-2}^+(z) - z(m_i(z)P_{2i-2}^+(z))' + P_{2i-3}^{+'}(z).$$

Substituting $z = 0$ and using the equality $P_{2i-2}^+(0) = 1$, one obtains

$$m_i(0) = P_{2i-3}^{+'}(0) - P_{2i-1}^{+'}(0).$$

Hence

$$\sum_{i=1}^N m_i(0) = P_{-1}^{+'}(0) - P_1^{+'}(0) + \cdots + P_{2N-3}^{+'}(0) - P_{2N-1}^{+'}(0) = -P_{2N-1}^{+'}(0).$$

This proves (3.11). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

3.2. **The class $\mathcal{U}_\kappa^k(J)$ and linear fractional transformations.** Let J and $\mathcal{Z}$ be the 2×2 matrices

$$J = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{Z} = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.$$

Definition 3.6. Let $W(z)$ be a 2×2 matrix valued function meromorphic in $\mathbb{C}_+$ and let $\mathfrak{h}_W^+$ be the domain of holomorphy of W in $\mathbb{C}_+$, $\kappa \in \mathbb{Z}_+$. Then $W(z)$ is called a *generalized J-inner* matrix valued function from the class $\mathcal{U}_\kappa(J)$, if

(i) the kernel

$$(3.13) \qquad \mathsf{K}_\omega^W(z) = \frac{J - W(z)JW(\omega)^*}{-i(z - \bar{\omega})}$$

has κ negative squares in $\mathfrak{h}_W^+ \times \mathfrak{h}_W^+$;

(ii) $J - W(\mu)JW(\mu)^* = 0$ for a.e. $\mu \in \mathbb{R}$.

A matrix valued function $W \in \mathcal{U}_\kappa(J)$ is said to belong to the class $\mathcal{U}_\kappa^k(J)$, $\kappa, k \in \mathbb{Z}_+$, if

$$(3.14) \qquad \mathcal{Z}W\mathcal{Z}^{-1} \in \mathcal{U}_k(J).$$

Consider the linear fractional transformation

$$(3.15) \qquad T_W[\tau] = (w_{11}\tau(z) + w_{12})(w_{21}\tau(z) + w_{22})^{-1}$$

associated with the matrix valued function $W(z) = (w_{i,j}(z))_{i,j=1}^2$. The linear fractional transformation associated with the product W_1W_2 of two matrix valued functions $W_1(z)$ and $W_2(z)$, coincides with the composition $T_{W_1} \circ T_{W_2}$.

The following statement is an easy corollary of Definition 3.6.

Lemma 3.7. *Let $W \in \mathcal{U}_{\kappa_1}^{k_1}(J)$ and $\tau \in \mathbf{N}_{\kappa_2}^{k_2}$, $\kappa_1, \kappa_2, k_1, k_2 \in \mathbb{Z}_+$. Then $T_W[\tau] \in \mathbf{N}_{\kappa'}^{k'}$, where $\kappa' \leq \kappa_1 + \kappa_2$, $k' \leq k_1 + k_2$.*

Proof. The proof of the inclusion $T_W[\tau] \in \mathbf{N}_{\kappa'}$ for some $\kappa' \le \kappa_1 + \kappa_2$ is similar to that in [13, Lemma 3.4].

Let us denote $f(z) = T_W(z)[\tau(z)]$. Then

$$zf(z) = T_{\mathcal{Z}W(z)\mathcal{Z}^{-1}}[z\tau(z)].$$

Since $\mathcal{Z}W\mathcal{Z}^{-1} \in \mathcal{U}_{k_1}$ and $z\tau \in \mathbf{N}_{k_2}$ then $f \in \mathbf{N}_{k'}$ for some $k' \le k_1 + k_2$. This proves that $T_W[\tau] \in \mathbf{N}_{\kappa'}^{k'}$ for some $\kappa' \le \kappa_1 + \kappa_2$, $k' \le k_1 + k_2$. $\qquad\square$

In the present paper two special types of matrix valued functions from $\mathcal{U}_\kappa^k(J)$ play an important role (see [17, Lemma 2.11, Lemma 2.12]).

Lemma 3.8. *Let $m(z)$ be a real polynomial such that $\kappa_-(zm(z)) = \kappa_1$, $\kappa_-(m(z)) = k_1$, let $M(z)$ be a 2×2 matrix valued function*

$$(3.16) \qquad M(z) = \begin{pmatrix} 1 & 0 \\ -zm(z) & 1 \end{pmatrix}$$

and let τ be a meromorphic function, such that

$$(3.17) \qquad \tau(z)^{-1} = o(z) \quad as \quad z \widehat{\to} \infty.$$

Then $M \in \mathcal{U}_{\kappa_1}^{k_1}(J)$ and the following equivalences hold:

$$(3.18) \qquad \tau \in \mathbf{N}_{\kappa_2} \iff T_M[\tau] \in \mathbf{N}_{\kappa_1+\kappa_2},$$

$$(3.19) \qquad \tau \in \mathbf{N}_{\kappa_2}^{k_2} \iff T_M[\tau] \in \mathbf{N}_{\kappa_1+\kappa_2}^{k_1+k_2}.$$

Lemma 3.9. *Let $l(z)$ be a real polynomial such that $\kappa_-(l(z)) = \kappa_1$, $\kappa_-(zl(z)) = k_1$, let $L(z)$ be the 2×2 matrix valued function*

$$(3.20) \qquad L(z) = \begin{pmatrix} 1 & l(z) \\ 0 & 1 \end{pmatrix}$$

and let ϕ be a meromorphic function, such that

$$(3.21) \qquad \phi(z) = o(1) \quad as \quad z \widehat{\to} \infty.$$

Then $L \in \mathcal{U}_{\kappa_1}^{k_1}(J)$ and the following equivalences hold:

$$(3.22) \qquad \phi \in \mathbf{N}_{\kappa_2} \iff T_L[\phi] \in \mathbf{N}_{\kappa_1+\kappa_2},$$

$$(3.23) \qquad \phi \in \mathbf{N}_{\kappa_2}^{k_2} \iff T_L[\phi] \in \mathbf{N}_{\kappa_1+\kappa_2}^{k_1+k_2}.$$

3.3. **Truncated indefinite moment problem and resolvent matrices.** As was mentioned in Theorem 1.2 the matrix valued function $W_{2N}(z)$ defined by (1.26) is the resolvent matrix of a truncated even indefinite moment problem $MP_\kappa^k(\mathbf{s}, 2N-1)$ with $\kappa \ge \kappa_N$ and $k \ge k_N^+$, where κ_N and k_N^+ are defined by (1.18). The resolvent matrix $W_{2N}(z)$ provides an example of generalized J−inner matrix valued function from the class $\mathcal{U}_{\kappa_N}^{k_N^+}(J)$.

Theorem 3.10. *([17]) Let $\mathbf{s} = \{s_i\}_{i=0}^\infty \in \mathcal{H}_\kappa^{k,reg}$, let $P_j^+(z)$ and $Q_j^+(z)$ be generalized Stieltjes polynomials of the first and second kind, let $m_i(z)$, l_i be defined by (2.15) and (2.16) and let $W_{2N}(z)$, $M_i(z)$ and L_i be given by (1.26) and (1.28). Then*

(i) *The matrix valued function $W_{2N}(z)$ admits the factorization*

$$(3.24) \qquad W_{2N}(z) = M_1(z)L_1 \dots M_N(z)L_N;$$

(ii) $W_{2N} \in \mathcal{U}_{\kappa_N}^{k_N^+}(J)$.

Proof. It follows from (3.1) that

(3.25)
$$W_{2N}(z) = W_{2N-2}(z)M_N(z)L_N.$$

Applying (3.25) N times one obtains (3.24).

By [17, Theorem 4.3 (4)]

$$\kappa_N = \sum_{j=1}^{N} \kappa_-(zm_j), \quad k_N^+ = \sum_{j=1}^{N} k_-(m_j) + \sum_{j=1}^{N} \kappa_-(zl_j).$$

Since by Lemmas 3.8, 3.9 $M_i \in \mathcal{U}_{\kappa_-(zm_i)}^{\kappa_-(m_i)}$, $L_i \in \mathcal{U}_0^{\kappa_-(zl_i)}$, $i \in \mathbb{N}$, one obtains from the factorization formula (3.24) $W_{2N} \in \mathcal{U}_{\kappa'}^{k'}(J)$, where

(3.26)
$$\kappa' \leq \kappa_N, \quad k' \leq k_N^+.$$

On the other hand it follows from Lemma 3.7 that $f := T_{W_{2N}[0]} \in \mathbf{N}_{\kappa_N}^{k_N^+}$. Therefore, by Lemma 3.7

(3.27)
$$\kappa_N \leq \kappa', \quad k_N^+ \leq k'.$$

The statement (ii) follows from (3.26) and (3.27). □

Let us formulate an analog of Theorems 1.2 and 3.10 for the odd truncated indefinite moment problem (see [17]).

Theorem 3.11. *Let* $\mathbf{s} = \{s_i\}_{i=0}^{2n_N-2} \in \mathcal{H}_\kappa^{k,reg}$, *let* $P_j^+(z)$ *and* $Q_j^+(z)$ $(0 \leq j \leq 2N-1)$ *be generalized Stieltjes polynomials, let* $M_i(z)$ *and* L_i *be given by* (1.28) *and let the matrix valued function* W_{2N-1} *be defined by*

(3.28)
$$W_{2N-1}(z) = \begin{pmatrix} Q_{2N-1}^+(z) & Q_{2N-2}^+(z) \\ P_{2N-1}^+(z) & P_{2N-2}^+(z) \end{pmatrix}.$$

Then

(i) *The odd moment problem* $MP_\kappa^k(\mathbf{s}, 2n_N - 2)$ *is solvable, if and only if*

(3.29)
$$\kappa_N := \nu_-(S_{n_N}) \leq \kappa \quad and \quad k_N := \nu_-(S_{n_N-1}^+) \leq k.$$

(ii) $f \in \mathcal{M}_\kappa^k(\mathbf{s}, 2n_N - 2)$ *if and only if* f *admits the representation*

(3.30)
$$f(z) = T_{W_{2N-1}(z)}[\tau(z)] = \frac{Q_{2N-1}^+(z)\tau(z) + Q_{2N-2}^+(z)}{P_{2N-1}^+(z)\tau(z) + P_{2N-2}^+(z)},$$

where $\tau \in N_{\kappa-\kappa_N}^{k-k_N}$ *and* $\tau(z)^{-1} = o(z)$ *as* $z\widehat{\to}\infty$.

(iii) *The matrix valued function* $W_{2N-1}(z)$ *belongs to the class* $\mathcal{U}_{\kappa_N}^{k_N}(J)$ *and admits the factorization*

(3.31)
$$W_{2N-1}(z) = M_1(z)L_1 \ldots L_{N-1}M_N(z).$$

Substituting in (3.30) $\tau(z) = \infty$ one obtains from Theorem 3.11 the following:

Corollary 3.12. *The function* $\frac{Q_{2N-1}^+(z)}{P_{2N-1}^+(z)}$ *belongs to the class* $\mathbf{N}_{\kappa_N}^{k_N}$.

Similarly, substituting in (1.23) $\tau(z) \equiv 0$ one obtains from Theorem 1.2.

Corollary 3.13. *The function* $\frac{Q_{2N}^+(z)}{P_{2N}^+(z)}$ *belongs to the class* $\mathbf{N}_{\kappa_N}^{k_N^+}$.

3.4. **Two lemmas about linear fractional transformations T_M and T_L.** The statements of Lemmas 3.8 and 3.9 fail to hold if the conditions

(3.32) $\tau(z)^{-1} = o(z) \quad \text{as} \quad z \widehat{\to} \infty,$

(3.33) $\phi(z) = o(1) \quad \text{as} \quad z \widehat{\to} \infty$

are not satisfied. In these cases the number of moments interpolating by the linear fractional transformations $T_W[\tau]$ can be reduced and also their indices can decrease. We will start with the linear fractional transformations T_M in the simplest cases when m is a positive constant and hence $M \in \mathcal{U}_0^0(J)$.

Lemma 3.14. *Let $M(z)$ be a 2×2 matrix valued function*

(3.34) $M(z) = \begin{pmatrix} 1 & 0 \\ -zm & 1 \end{pmatrix}, \quad m > 0,$

let $\tau \in \mathbf{N}_\kappa^k$ be a function from $\mathbf{N}_\kappa^k$, such that $\tau(z)^{-1} \neq o(z)$ as $z \widehat{\to} \infty$ and let $\phi = T_M[\tau]$. Then

  (i) *either $\phi \in \mathbf{N}_\kappa^k$ and $\phi(z) = o(1)$ as $z \widehat{\to} \infty$,*
  (ii) *or $\phi \in \mathbf{N}_{\kappa-1}^k$.*

Proof. **1.** *Verification of* (i) *for $\tau \in \mathbf{N}_\kappa^k$, such that:*

(3.35) $\lim_{z \widehat{\to} \infty} \frac{-1}{z\tau(z)} = -\infty.$

By Lemma 2.1 $-\tau^{-1} \in \mathbf{N}_\kappa^{-k}$. If (3.35) holds then $mz - \tau(z)^{-1}$ has GPNT (generalized pole of negative type) at ∞ of the same multiplicity as $-\tau(z)^{-1}$, i.e.

(3.36) $\kappa_\infty \left(mz - \frac{1}{\tau(z)} \right) = \kappa_\infty \left(-\frac{1}{\tau(z)} \right).$

Since also

(3.37) $\kappa_\infty \left(m - \frac{1}{z\tau(z)} \right) = \kappa_\infty \left(-\frac{1}{z\tau(z)} \right),$

one obtains by Theorem 2.2 $mz - \tau(z)^{-1} \in \mathbf{N}_\kappa^{-k}$. Hence $\phi(z) = \frac{-1}{mz - \tau(z)^{-1}} \in \mathbf{N}_\kappa^k$ by Lemma 2.1 and, moreover, $\phi(z) = o(1)$ as $z \widehat{\to} \infty$, since

$$\lim_{z \widehat{\to} \infty} \phi(z) = \lim_{z \widehat{\to} \infty} \frac{1/z}{m - (z\tau(z))^{-1}} = 0.$$

  2. *Verification of* (i) *for $\tau \in \mathbf{N}_\kappa^k$, such that:*

(3.38) $\lim_{z \widehat{\to} \infty} \frac{-1}{z\tau(z)} = a$

and $a \geq 0$. In this case $\kappa_\infty(-\tau(z)^{-1}) = 0$ and $-\tau(z)^{-1}$ admits the representation

(3.39) $-\tau(z)^{-1} = az - \tau_1(z)^{-1},$

where $\lim_{z \widehat{\to} \infty} \frac{-1}{z\tau_1(z)} = 0$. Then the function $mz - \tau(z)^{-1} = (m+a)z - \tau_1(z)^{-1}$ has no GPNT at ∞ since

$$\lim_{z \widehat{\to} \infty} \left(m - \frac{1}{z\tau(z)} \right) = m + a > 0.$$

Then (3.36) and (3.37) hold and by Theorem 2.2 $mz - \tau(z)^{-1} \in \mathbf{N}_\kappa^{-k}$. Moreover,

(3.40) $\lim_{z \widehat{\to} \infty} \phi(z) = \lim_{z \widehat{\to} \infty} \frac{1/z}{m + a - (z\tau(z))^{-1}} = 0.$

and hence (i) holds.

3. *Verification of* (i) *for* $\tau \in \mathbf{N}_\kappa^k$, *such that* (3.38) *holds and* $a < -m$: In this case

$$\lim_{z \widehat{\to} \infty} \left(m - \frac{1}{z\tau(z)} \right) = m + a < 0$$

and hence

$$\kappa_\infty \left(mz - \frac{1}{\tau(z)} \right) = \kappa_\infty \left(-\frac{1}{\tau(z)} \right) = 1.$$

Since also (3.37) holds then by Theorem 2.2 $mz - \tau(z)^{-1} \in \mathbf{N}_\kappa^{-k}$, and by (3.40) $\phi(z) = o(1)$ as $z \widehat{\to} \infty$. Hence (i) holds.

4. *Verification of* (ii) *for* $\tau \in \mathbf{N}_\kappa^k$, *such that* (3.38) *holds and* $a \in [-m, 0)$: In this case

$$\kappa_\infty \left(mz - \frac{1}{\tau(z)} \right) = 0, \quad \kappa_\infty \left(-\frac{1}{\tau(z)} \right) = 1$$

and since (3.37) holds then $mz - \tau(z)^{-1} \in \mathbf{N}_{\kappa-1}^{-k}$ by Theorem 2.2. Therefore, $\phi(z) = \frac{-1}{mz - \tau(z)^{-1}} \in \mathbf{N}_{\kappa-1}^k$ by Lemma 2.1. This proves (ii). $\qquad \square$

Lemma 3.15. *Let* L *be the* 2×2 *matrix*

$$(3.41) \qquad L = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}, \quad where \quad l > 0,$$

let ϕ *be a function from* $\mathbf{N}_\kappa^k$, *such that*

$$(3.42) \qquad \phi(z) \neq o(1) \quad as \quad z \widehat{\to} \infty$$

and let $\tau = T_L[\phi] = l + \phi$. *Then*
   (i) *either* $\tau \in \mathbf{N}_\kappa^k$ *and* $\tau(z)^{-1} = o(z)$ *as* $z \widehat{\to} \infty$,
   (ii) *or* $\tau \in \mathbf{N}_\kappa^{k-1}$.

Proof. **1.** *Verification of* (i) *for* $\phi \in \mathbf{N}_\kappa^k$, *such that* $z\phi(z)$ *has a GPNT at* ∞ *and*:

$$(3.43) \qquad \lim_{z \widehat{\to} \infty} \phi(z) = -\infty.$$

The function $z\tau(z) = lz + z\phi(z)$ has GPNT at ∞ of the same multiplicity as $z\phi(z)$ and by Theorem 2.2 $z\tau(z) \in \mathbf{N}_k$. Hence $\tau \in \mathbf{N}_\kappa^k$ and the equality

$$\lim_{z \widehat{\to} \infty} \frac{1}{z\tau(z)} = \lim_{z \widehat{\to} \infty} \frac{1}{lz + z\phi(z)} = 0$$

implies that $\tau(z)^{-1} = o(z)$ as $z \widehat{\to} \infty$. Thus (1) holds.

2. *Verification of* (i) *for* $\phi \in \mathbf{N}_\kappa^k$, *such that* $\lim_{z \widehat{\to} \infty} \phi(z) = a$ *and* $a > 0$: In this case $z\phi(z)$ admits the representation

$$(3.44) \qquad z\phi(z) = az + z\phi_1(z),$$

where $\lim_{z \widehat{\to} \infty} \phi_1(z) = 0$ and $\phi_1 \in \mathbf{N}_\kappa^k$ by Theorem 2.2. The function $z\tau(z) = lz + z\phi(z)$ has no GPNT at ∞ and therefore $\tau \in \mathbf{N}_\kappa^k$. Moreover,

$$(3.45) \qquad \lim_{z \widehat{\to} \infty} \frac{1}{z\tau(z)} = \lim_{z \widehat{\to} \infty} \frac{1}{(l+a)z + z\phi(z)} = 0.$$

and hence (i) holds.

3. *If* $\phi \in \mathbf{N}_\kappa^k$, $\lim_{z \widehat{\to} \infty} \phi(z) = a$ *and* $a < -l$, *then also* (i) *holds*: In this case

$$\kappa_\infty(z\phi) = 1 \quad \text{and} \quad \kappa_\infty(z\tau) = 1,$$

that is the functions $z\phi(z)$ and $z\tau(z)$ have GPNT at ∞ of the same multiplicity. By Theorem 2.2 $\tau \in \mathbf{N}_\kappa^k$ and by (3.45) $\tau(z)^{-1} = o(z)$ as $z \widehat{\to} \infty$. Hence (i) holds.

4. *If* $\lim_{z \widehat{\to} \infty} \phi(z) = a$ *and* $a \in [-l, 0)$ *then* (ii) *holds*: In this case

$$\kappa_\infty(z\phi) = 1 \quad \text{and} \quad \kappa_\infty(z\tau) = 0,$$

and hence $z\tau \in \mathbf{N}_{k-1}$ by Theorem 2.2. Thus $\tau \in \mathbf{N}_\kappa^{k-1}$. □

Corollary 3.16. *Let a matrix valued function* $W_{2n}(z)$ *of the form* (1.26) *belong to the class* $\mathcal{U}_0^0(J)$, *let* $\phi \in \mathbf{N}_\kappa^k$, (3.33) *fails to hold, let* $n \geq \min\{\kappa, k\}$ *and let* $f = T_{W_{2n}}[\phi]$. *Then there exists* $r \in \mathbb{Z}_+$ ($r \leq \min\{\kappa, k\}$), *such that*

(i) *either* $f \in \mathcal{M}_{\kappa-r}^{k-r}(\mathbf{s}, 2(n-r) - 2)$;

(ii) *or* $f \in \mathcal{M}_{\kappa-r}^{k-r-1}(\mathbf{s}, 2(n-r) - 3)$.

In particular, $f(z) = o(1)$ *as* $z \widehat{\to} \infty$.

Proof. By Theorem 3.11 the matrix valued function $W_{2n}(z)$ admits the factorization

$$(3.46) \qquad\qquad W_{2n}(z) = M_1(z)L_1 \ldots M_n(z)L_n.$$

Let us denote $\tau_1 := T_{L_n}[\phi]$. By Lemma 3.15 there are three possibilities

(a1) either $\tau_1 \in \mathbf{N}_\kappa^k$ and $\tau_1^{-1}(z) = o(z)$;

(a2) or $\tau_1 \in \mathbf{N}_\kappa^{k-1}$ and $\tau_1^{-1}(z) = o(z)$;

(a3) or $\tau_1 \in \mathbf{N}_\kappa^{k-1}$ and $\tau_1^{-1}(z) \neq o(z)$.

In the case (a1) one obtains by Theorem 3.11 $f \in \mathcal{M}_\kappa^k(\mathbf{s}, 2n - 2)$, which gives (i) with $r = 0$.

In the case (a2) by Theorem 3.11 $f \in \mathcal{M}_\kappa^{k-1}(\mathbf{s}, 2n-2) \subset \mathcal{M}_\kappa^{k-1}(\mathbf{s}, 2n-3)$, which gives (ii) with $r = 0$.

In the case (a3) we apply Lemma 3.14 to the function $\phi_1(z) = T_{M_n}[\tau_1(z)]$ and then again there are three possibilities:

(b1) $\phi_1 := T_{M_n}[\tau] \in \mathbf{N}_\kappa^{k-1}$ and $\phi_1(z) = o(1)$;

(b2) $\phi_1 := T_{M_n}[\tau] \in \mathbf{N}_{\kappa-1}^{k-1}$ and $\phi_1(z) = o(1)$;

(b3) $\phi_1 := T_{M_n}[\tau] \in \mathbf{N}_{\kappa-1}^{k-1}$ and $\phi_1(z) \neq o(1)$.

In the case (b1) by Theorem 1.2 one obtains $f \in \mathcal{M}_\kappa^{k-1}(\mathbf{s}, 2n-3)$ which gives (2) with $r = 0$.

In the case (b2) by Theorem 1.2 $f \in \mathcal{M}_{\kappa-1}^{k-1}(\mathbf{s}, 2n - 3) \subset \mathcal{M}_{\kappa-1}^{k-1}(\mathbf{s}, 2n - 4)$ which gives (i) with $r = 1$.

In the case (b3) one should continue this process based on Lemma 3.14, Lemma 3.15, Theorem 1.2 and Theorem 3.11.

If, for instance, $k \leq \kappa$ and if the process will not stop until the step $r = k$, then when applying Lemma 3.15 to $\phi_k \in \mathbf{N}_{\kappa-k}^0$ one obtains only one possibility: $\tau_{k+1} \in \mathbf{N}_{\kappa-k}^0$ and $\tau_{k+1}(z)^{-1} = o(z)$ as $z \widehat{\to} \infty$. Then the process stops and $f \in \mathcal{M}_{\kappa-k}^0(\mathbf{s}, 2(n-k) - 2)$ by Theorem 3.11.

Similarly, one can treat the case $\kappa < k$ by using Lemma 3.14 and Theorem 1.2. □

4. FULL INDEFINITE MOMENT PROBLEM $MP_\kappa^k(\mathbf{s})$

4.1. **Indefinite moment problem** $MP_\kappa^k(\mathbf{s})$ **for** $\mathbf{s} \in \mathcal{H}_0^0$. Recall, that the moment problem $MP_\kappa^k(\mathbf{s})$ is called *indeterminate* if it has more then one solution. A sequence $\mathbf{s}$ is called *nondegenerate*, if

$$(4.1) \qquad \text{there is } N \in \mathbb{N}, \quad \text{such that} \quad D_n \neq 0, \quad D_n^+ \neq 0 \quad \text{for all} \quad n \geq N.$$

In this subsection we consider the indefinite moment problem $MP_\kappa^k(\mathbf{s})$ for a nondegenerate sequence $\mathbf{s} \in \mathcal{H}_0^0$. As is known, see [1, Theorem 0.5], for a nondegenerate sequence $\mathbf{s} \in \mathcal{H}_0^0$ the corresponding moment problem $MP_0^0(\mathbf{s})$ is indeterminate, if and only if

$$(4.2) \qquad\qquad \sum_{i=1}^\infty m_i < \infty \quad \text{and} \quad \sum_{i=1}^\infty l_i < \infty.$$

Theorem 4.1. *Let s be a nondegenerate sequence from $\mathcal{H}_0^0$ and let (4.2) holds. Then*

   (i) *The moment problem $MP_\kappa^k(s)$ is solvable and indeterminate for any pair of $\kappa, k \in \mathbb{Z}_+$.*

   (ii) *The sequence of resolvent matrices $W_{2n}(z)$ converges to an entire matrix valued function $W_\infty^+(z) = (w_{ij}^+(z))_{i,j=1}^2$ of order $\leq 1/2$.*

   (iii) *The formula*

$$(4.3) \qquad f(z) = \frac{w_{11}^+(z)\tau(z) + w_{12}^+(z)}{w_{21}^+(z)\tau(z) + w_{22}^+(z)}$$

establishes a one-to-one correspondence between the class $\mathcal{M}_\kappa^k(s)$ and the set of functions $\tau \in \mathbf{N}_\kappa^k$.

Proof. **1.** *Verification of* (ii): Due to [27, Appendix II.13] the Stieltjes moment problem $\mathcal{M}_0^0(\mathbf{s}^{(N)})$ is indeterminate if and only if (4.2) holds and in this case the sequence $W_{2N}(z)$ (see (1.26)) of its resolvent matrices converges to an entire matrix valued function $W_\infty^+(z) = (w_{ij}^+(z))_{i,j=1}^2$ of order $\leq 1/2$.

2. *Verification of the implication $f \in \mathcal{M}_\kappa^k(s) \implies \tau \in \mathbf{N}_\kappa^k$*: Since $f \in \mathcal{M}_\kappa^k(\mathbf{s})$ then $f \in \mathcal{M}_\kappa^k(\mathbf{s}, 2n-1)$ for every $n \in \mathbb{N}$. By Theorem 1.2 there exists a sequence of functions $\phi_N(z) \in \mathbf{N}_\kappa^k$ such that (3.33) holds and

$$(4.4) \qquad f(z) = T_{W_{2N}(z)}[\phi_N(z)] \quad \text{for every} \quad N \in \mathbb{N}.$$

It follows from (4.4) that

$$-(Q_{2N-1}^+(z) - f(z)P_{2N-1}^+(z))\phi_N(z) = Q_{2N}^+(z) - f(z)P_{2N}^+(z).$$

Notice that $Q_{2N-1}^+(z) - f(z)P_{2N-1}^+(z) \not\equiv 0$, since otherwise we would have $Q_{2N}^+(z) - f(z)P_{2N}^+(z) \equiv 0$, which contradicts the generalized Liouville-Ostrogradsky identity (3.7).

As was mentioned above the matrix valued functions $W_{2N}(z)$ converges locally uniformly in $\mathbb{C}$ to $W_\infty^+(z) = (w_{ij}^+(z))_{i,j=1}^2$, in particular,

$$w_{11}^+(z) = \lim_{N\to\infty} Q_{2N-1}^+(z), \quad w_{21}^+(z) = \lim_{N\to\infty} P_{2N-1}^+(z).$$

By Corollary 3.12 $\frac{Q_{2N-1}^+}{P_{2N-1}^+} \in \mathbf{N}_0^0 = S$. Hence the limiting function $\frac{w_{11}^+(z)}{w_{21}^+(z)}$ belongs to $\mathbf{N}_0^0 = S$ and it corresponds to $\phi = \infty$ in the linear fractional transformation $f = T_{W_\infty^+}[\tau]$. Apart from this case the function

$$(4.5) \qquad w_{11}^+(z) - f(z)w_{21}^+(z) = \lim_{N\to\infty}(Q_{2N-1}^+(z) - f(z)P_{2N-1}^+(z))$$

is not identically equal to 0. Let Ω be the open set of points in $\mathbb{C}_+$ such that $w_{11}^+(z) - f(z)w_{21}^+(z) \neq 0$. Then for every point $z \in \Omega$ the sequence of functions

$$\phi_N(z) := \frac{Q_{2N}^+(z) - f(z)P_{2N}^+(z)}{Q_{2N-1}^+(z) - f(z)P_{2N-1}^+(z)}$$

is correctly defined in a neighborhood of z and converges locally uniformly in Ω to a function $\phi(z)$. Since $\phi_N \in \mathbf{N}_\kappa^k$ then $\phi \in \mathbf{N}_{\kappa'}^{k'}$ with $\kappa' \leq \kappa$ and $k' \leq k$. It follows from (4.4) that

$$(4.6) \qquad f(z) = T_{W_\infty^+(z)}[\phi(z)]$$

and by Lemma 3.7 $f \in \mathbf{N}_{\kappa''}^{k''}$ with $\kappa'' \leq \kappa' \leq \kappa$ and $k'' \leq k' \leq k$. Since $f \in \mathbf{N}_\kappa^k$ this implies $\kappa'' = \kappa' = \kappa$ and $k'' = k' = k$ and hence $\tau \in \mathbf{N}_\kappa^k$.

3. *Proof of the fact that $T_{W_\infty^+}[\tau]$ satisfies (3.33) for every $\tau \in \mathbf{N}_\kappa^k$, $\kappa, k \in \mathbb{Z}_+$.* Application of the Schur algorithm to the Stieltjes moment problem $\mathcal{M}_\kappa^k(\mathbf{s})$ gives on the N−th step an induced sequence $\mathbf{s}^{(N)}$, which can be found as the sequence of coefficients

of the series expansion $-\sum\limits_{i=0}^{\infty} s_i^{(N)} z^{-(i+1)}$ corresponding to the continued fraction (2.17).

Denote by $W_{2j}^{(N)}(z)$ the resolvent matrix of the corresponding Stieltjes moment problem $\mathcal{M}_0^0(\mathbf{s}^{(N)}, 2j-1)$.

$$(4.7) \qquad W_{2j}^{(N)}(z) = \begin{pmatrix} Q_{2j-1}^{N,+}(z) & Q_{2j}^{N,+}(z) \\ P_{2j-1}^{N,+}(z) & P_{2j}^{N,+}(z) \end{pmatrix}, \quad j \in \mathbb{N}.$$

Then for every $j \in \mathbb{N}$ the resolvent matrix $W_{2j}(z)$ admits the factorization (see [17, Proposition 4.7])

$$(4.8) \qquad W_{2j}(z) = W_{2N}(z) W_{2(j-N)}^{(N)}(z).$$

Taking the limit as $j \to \infty$ one obtains the following factorization of the entire matrix valued function $W_\infty^+(z)$

$$(4.9) \qquad W_\infty^+(z) = W_{2N}(z) W_\infty^{(N)}(z),$$

where $W_\infty^{(N)}(z)$ is the resolvent matrix of the induced Stieltjes moment problem $\mathcal{M}_0^0(\mathbf{s}^{(N)})$. Then $f = T_{W_\infty^+}[\tau]$ admits the representation

$$(4.10) \qquad f = T_{W_{2N}}[\phi_N], \quad \text{where} \quad \phi_N = T_{W_\infty^{(N)}}[\tau] \in \mathbf{N}_{\kappa'}^{k'}, \quad \text{with } \kappa' \le \kappa \text{ and } k' \le k.$$

Since $W_{2N} \in \mathcal{U}_0^0$ by Corollary 3.16 one obtains $f = o(1)$, if $N > \min\{\kappa, k\}$.

4. *Verification of the implication* $\tau \in \mathbf{N}_\kappa^k \implies f = T_{W_\infty^+}[\tau] \in \mathcal{M}_\kappa^k(\mathbf{s})$: Due to the above item **3** the function ϕ_N from (4.10) satisfies the condition (3.33).

Next By Theorem 1.2 $f \in \mathcal{M}_{\kappa'}^{k'}(\mathbf{s}, 2N-1)$. Since N can be chosen arbitrarily large $f \in \mathcal{M}_{\kappa'}^{k'}(\mathbf{s})$. Now it follows from the item **2** that $\tau \in \mathbf{N}_{\kappa'}^{k'}$ and hence $\kappa' = \kappa$ and $k' = k$.

5. *Verification of* (i): Solvability of the indefinite Stieltjes moment problem $MP_\kappa^k(\mathbf{s})$ for any pair of $\kappa, k \in \mathbb{Z}_+$ follows from the item **4**. The formula (4.3) gives two different solutions of the problem $MP_\kappa^k(\mathbf{s})$ for different parameter functions $\tau_1, \tau_2 \in \mathbf{N}_\kappa^k$ and thus the problem $MP_\kappa^k(\mathbf{s})$ is indeterminate. $\qquad\square$

4.2. Indefinite moment problem $MP_\kappa^k(\mathbf{s})$, general case.

Theorem 4.2. *Let $\mathbf{s}$ be a nondegenerate sequence from $\mathcal{H}_{\kappa_0}^{k_0, reg}$, $\kappa_0, k_0 \in \mathbb{N}$ and let l_j and $m_j(z)$ $(j \in \mathbb{N})$ be parameters of the generalized $\mathbf{S}-$fraction (2.12). Then the moment problem $MP_{\kappa_0}^{k_0}(\mathbf{s})$ is indeterminate, if and only if*

$$(4.11) \qquad \sum_{i=1}^{\infty} m_i(0) < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} l_i < \infty.$$

If (4.11) *holds, then*

  (i) *The sequence of resolvent matrices $W_{2n}(z)$ converges to an entire matrix valued function $W_\infty^+(z) = (w_{ij}^+(z))_{i,j=1}^2$ of order $\le 1/2$.*
  (ii) *The moment problem $MP_\kappa^k(\mathbf{s})$ is solvable, if and only if*

$$(4.12) \qquad \kappa_0 \le \kappa, \quad \text{and} \quad k_0 \le k.$$

  (iii) *The formula*

$$(4.13) \qquad f(z) = \frac{w_{11}^+(z)\tau(z) + w_{12}^+(z)}{w_{21}^+(z)\tau(z) + w_{22}^+(z)}$$

establishes a one-to-one correspondence between the class $\mathcal{M}_\kappa^k(\mathbf{s})$ and the set of functions $\tau \in \mathbf{N}_{\kappa-\kappa_0}^{k-k_0}$.

Proof. **1.** *Redaction of the indefinite moment problem* $MP_\kappa^k(\boldsymbol{s})$ *to a classical one:* Let us choose N big enough, so that

$$(4.14) \qquad \nu_-(S_j) = \kappa_0 = \nu_-(S_{n_N}), \quad \nu_-(S_j^{(1)}) = k_0 = \nu_-(S_{n_N}^{(1)}) \quad \text{for all} \quad j \geq n_N.$$

Then the induced sequence $\mathbf{s}^{(N)}$ which arises on the $N-$th step of the Schur algorithm (see Section 2.2) belongs to the class $\mathcal{H}_0^0$. The corresponding Stieltjes moment problem $\mathcal{M}_0^0(\mathbf{s}^{(N)})$ is classical. By Lemma 2.4

$$(4.15) \qquad f \in \mathcal{M}_\kappa^k(\mathbf{s}) \iff f_N = T_{W_{2n_N}}[f] \in \mathcal{M}_{\kappa-\kappa_0}^{k-k_0}(\mathbf{s}^{(N)}).$$

In particular,

$$(4.16) \qquad f \in \mathcal{M}_{\kappa_0}^{k_0}(\mathbf{s}) \iff f_N = T_{W_{2n_N}}[f] \in \mathcal{M}_0^0(\mathbf{s}^{(N)}).$$

Hence the problem $\mathcal{M}_{\kappa_0}^{k_0}(\mathbf{s}^{(N)})$ is indeterminate if and only if the Stieltjes moment problem $\mathcal{M}_0^0(\mathbf{s}^{(N)})$ is indeterminate. The latter is equivalent to (4.11).

2. *Verification of* (i): Let $P_j^{N,+}(z)$ and $Q_j^{N,+}(z)$ be generalized Stieltjes polynomials associated with the moment problem $\mathcal{M}_0^0(\mathbf{s}^{(N)})$ and let $W_{2j}^{(N)}$ (see (4.7)). If (4.11) holds, then the Stieltjes moment problem $\mathcal{M}_0^0(\mathbf{s}^{(N)})$ is indeterminate and the sequence $W_{2j}^{(N)}$ converges to an entire matrix valued function $W_\infty^{(N)}(z) = (w_{ij}^{(N)}(z))_{i,j=1}^2$ of order $\leq 1/2$.

The resolvent matrices $W_{2j}(z)$ of the indefinite moment problem $MP_{\kappa_0}^{k_0}(\mathbf{s}, 2j-1)$ are connected with the resolvent matrices $W_{2(j-N)}^{(N)}(z)$ of the induced moment problem $MP_0^0(\mathbf{s}^{(N)}, 2(j-N)-1)$ by the formula (4.8). Therefore, the sequence of matrix valued functions $W_{2n-1}(z)$ also converges to an entire matrix valued function $W_\infty^+(z) = (w_{ij}^+(z))_{i,j=1}^2$ of order $\leq 1/2$, which is connected with $W_\infty^{(N)}(z)$ by the formula (4.9).

3. *Verification of* (ii) *and* (iii): By Proposition 2.1 (v) the inequalities (4.12) are necessary for solvability of the problem $\mathcal{M}_\kappa^k(\mathbf{s})$.

Now assume that (4.12) holds, let $\tau \in \mathbf{N}_{\kappa-\kappa_0}^{k-k_0}$ and let $f = T_{W_\infty^+}[\tau]$. Then by Theorem 4.1

$$(4.17) \qquad f_N = T_{W_\infty^{(N)}}[\tau] \in \mathcal{M}_{\kappa-\kappa_0}^{k-k_0}(\mathbf{s}^{(N)}).$$

By (4.9) and Lemma 2.4

$$(4.18) \qquad f = T_{W_{2N}}[f_N] \in \mathcal{M}_\kappa^k(\mathbf{s}).$$

Conversely, let $f \in \mathcal{M}_\kappa^k(\mathbf{s})$. Then by Lemma 2.4 there is a function $f_N \in \mathcal{M}_{\kappa-\kappa_0}^{k-k_0}(\mathbf{s}^{(N)})$ such that $f = T_{W_{2N}}[f_N]$. Hence by Theorem 4.1 there exists a function $\tau \in \mathbf{N}_{\kappa-\kappa_0}^{k-k_0}$, such that $f_N = T_{W_\infty^{(N)}}[\tau]$. Therefore, $f = T_{W_\infty^+}[\tau]$ for $\tau \in \mathbf{N}_{\kappa-\kappa_0}^{k-k_0}$. $\qquad\square$

Proposition 4.3. *Let* $\boldsymbol{s} = \{s_j\}_{j=0}^\infty \in \mathcal{H}_\kappa^{k,reg}$, *let* $P_i(z)$ *and* $Q_i(z)$ $(i \in \mathbb{Z}_+)$ *be Lanczos polynomials of the first and second kind and let* l_j *and* $m_j(z)$ $(j \in \mathbb{N})$ *be parameters of the generalized* $\mathbf{S}-$*fraction* (2.12). *Assume additionally that*

$$(4.19) \qquad\qquad l_j > 0 \quad \text{for all} \quad j \in \mathbb{N}.$$

Then the following statements are equivalent:

(i) *the moment problem* $MP_\kappa(\boldsymbol{s})$ *is indeterminate;*
(ii) *the following series converge*

$$(4.20) \qquad \sum_{i=0}^\infty |P_i(0)|^2 \widetilde{b}_i^{-1} < \infty \quad \text{and} \quad \sum_{i=0}^\infty |Q_i(0)|^2 \widetilde{b}_i^{-1} < \infty;$$

(iii) *the following series converges*

$$(4.21) \qquad , \sum_{i=1}^{\infty} (l_1 + l_2 + \cdots + l_i)^2 d_{i+1}(0) < \infty.$$

Proof. By [8, 31] (i) and (ii) are equivalent.

Let us show, that (ii) and (iii) are equivalent. By the first equality in (3.10)

$$Q_i(0) = -(l_1 + l_2 + \cdots + l_i)P_i(0).$$

and hence by (3.11)

$$\sum_{i=0}^{N} |Q_i(0)|^2 \widetilde{b}_i^{-1} = \sum_{i=0}^{N} (l_1 + l_2 + \cdots + l_i)^2 |P_i(0)|^2 \widetilde{b}_i^{-1} = \sum_{i=1}^{N} (l_1 + l_2 + \cdots + l_i)^2 d_{i+1}(0).$$

This proves the implication (ii) $\Rightarrow$ (iii).

Conversely, if $\sum_{i=0}^{\infty} |Q_i(0)|^2 \widetilde{b}_i^{-1} < \infty$, then the convergence of the series $\sum_{i=0}^{\infty} |P_i(0)|^2 \widetilde{b}_i^{-1}$ follows from the inequality

$$l_1^2 \sum_{i=0}^{N} |P_i(0)|^2 \widetilde{b}_i^{-1} \leq \sum_{i=0}^{N} (l_1 + l_2 + \cdots + l_i)^2 |P_i(0)|^2 \widetilde{b}_i^{-1} = \sum_{i=0}^{\infty} |Q_i(0)|^2 \widetilde{b}_i^{-1}.$$

This proves the implication (iii) $\Rightarrow$ (ii). $\qquad\qquad\square$

Remark 4.4. In the case $\kappa = k = 0$ the above criterion for the Hamburger moment problem to be indeterminate is well known, see [1, Appendix, Theorem 0.5]. When treating system (3.1) as a Stieltjes string with masses m_j and lengths l_j one can consider series (4.21) as "the moment of inertia" of the string.

5. Padé approximants

Definition 5.1. ([3]) The $[L/M]$ Padé approximant for a formal power series

$$(5.1) \qquad -\sum_{j=0}^{\infty} \frac{s_j}{z^{j+1}}$$

is a ratio

$$(5.2) \qquad f^{[L/M]}(z) = \frac{A^{[L/M]}(1/z)}{B^{[L/M]}(1/z)}$$

of polynomials $A^{[L/M]}$, $B^{[L/M]}$ of formal degree L, M, respectively, such that $B^{[L/M]}(0) \neq 0$ and

$$(5.3) \qquad f^{[L/M]}(z) + \sum_{j=0}^{L+M} \frac{s_j}{z^{j+1}} = O\left(\frac{1}{z^{L+M+1}}\right), \quad z \widehat{\rightarrow} \infty.$$

The $[n/n]$ Padé approximant is called diagonal and the $[n/n-1]$ Padé approximant is called subdiagonal.

Remark 5.2. Notice, that for diagonal Padé approximants the representation (5.2) is equivalent to the representation

$$f^{[n/n]}(z) = \frac{z^n A^{[n/n]}(1/z)}{z^n B^{[n/n]}(1/z)}$$

as a ratio, where the numerator $z^n A^{[n/n]}(1/z)$ is a polynomial of formal degree n and the denominator $z^n B^{[n/n]}(1/z)$ is a polynomial of exact degree n. For subdiagonal Padé approximants the representation (5.2) is equivalent to the representation

$$f^{[n/n-1]}(z) = \frac{z^n A^{[n/n-1]}(1/z)}{z^n B^{[n/n-1]}(1/z)}$$

as a ratio, where the numerator is a polynomial of formal degree n and the denominator is a polynomial of exact degree n vanishing at 0.

Explicit formula for diagonal Padé approximants for sequences $\mathbf{s} = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}_\kappa$ was found in [7], in the classical case $\mathbf{s} \in \mathcal{H}_0$ see [3, 38]. In this section we will formulate the corresponding statements for sequences $\mathbf{s} = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}_\kappa^{k,reg}$.

Proposition 5.3. *Let $\mathbf{s} = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}_\kappa^{k,reg}$, $\kappa, k \in \mathbb{Z}_+$. Then the $[n/n]$ Padé approximant for a formal power series (5.1) exists if $n \in \mathcal{N}(\mathbf{s})$ and*

$$(5.4) \qquad f^{[n_j/n_j]}(z) = -\frac{Q_j(z)}{P_j(z)} = \frac{Q_{2j}^+(z)}{P_{2j}^+(z)}, \quad j \in \mathbb{N}.$$

Proof. We present a proof of this statement from [17] for the convenience of the reader. It follows from (3.3) and Theorem 3.11 that the function

$$-\frac{Q_j(z)}{P_j(z)} = \frac{Q_{2j}^+(z)}{P_{2j}^+(z)} = T_{W_{2j}(z)}[0]$$

belongs to $\mathcal{M}(\mathbf{s}, 2n_j - 1)$. Therefore, the function $-\frac{Q_j(z)}{P_j(z)}$ has the asymptotic

$$(5.5) \qquad -\frac{Q_j(z)}{P_j(z)} = -\frac{s_0}{z} - \cdots - \frac{s_{2n_j-1}}{z^{2n_j}} + O\left(\frac{1}{z^{2n_j+1}}\right), \quad z\widehat{\rightarrow}\infty.$$

Since $Q_j(z)$ is a polynomial of degree $n_j - n_1 < n_j$ and $P_j(z)$ is a polynomial of exact degree n_j the function $-\frac{Q_j(z)}{P_j(z)}$ is the $[n_j/n_j]$ Padé approximant for the formal power series (5.1) due to Remark 5.2 and (5.5). $\qquad \square$

In the following proposition stated in [16] without proof it is shown that the subdiagonal Padé approximants can be calculated in terms of generalized Stieltjes polynomials.

Proposition 5.4. *Let $\mathbf{s} = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}_\kappa^{k,reg}$, $\kappa, k \in \mathbb{Z}_+$. Then the $[n_j/n_j - 1]$ Padé approximants for the formal power series (5.1) exists and has the form*

$$(5.6) \qquad f^{[n_j/n_j-1]}(z) = \frac{Q_{2j-1}^+(z)}{P_{2j-1}^+(z)}, \quad j \in \mathbb{N}.$$

Proof. It follows from (3.3) that

$$\frac{Q_{2j-1}^+(z)}{P_{2j-1}^+(z)} = T_{W_{2j-1}(z)}[\infty].$$

By Theorem 3.11 the function $\frac{Q_{2j-1}^+(z)}{P_{2j-1}^+(z)}$ belongs to $\mathcal{M}(\mathbf{s}, 2n_j - 2)$, and hence it has the asymptotic

$$(5.7) \qquad \frac{Q_{2j-1}^+(z)}{P_{2j-1}^+(z)} = -\frac{s_0}{z} - \cdots - \frac{s_{2n_j-2}}{z^{2n_j-1}} + O\left(\frac{1}{z^{2n_j+1}}\right), \quad z\widehat{\rightarrow}\infty.$$

Since $Q_{2j-1}^+(z)$ is a polynomial of degree $n_j - n_1 < n_j$ and $P_{2j-1}^+(z)$ is a polynomial of exact degree n_j vanishing at 0 the function $-\frac{Q_{2j-1}^+(z)}{P_{2j-1}^+(z)}$ is the $[n_j/n_j-1]$ Padé approximant for the formal power series (5.1) due to Remark 5.2 and (5.7). $\qquad\square$

Lemma 5.5. *Let $\mathbf{s} = \{s_j\}_{j=0}^\infty \in \mathcal{H}_\kappa^{k,reg}$, $\kappa, k \in \mathbb{Z}_+$, let $W_{2N}(z)$ be given by (1.26), $N \in \mathbb{N}$ and let $\mathbf{s}^{(N)}$ be the induced sequence defined in Lemma 2.4, and let*

$$(5.8) \qquad -\sum_{j=0}^\infty s_j^{(N)} z^{-(j+1)}$$

be the corresponding formal power series. Then

   (i) *the diagonal Padé approximants for the formal power series (5.1) are connected with diagonal $g^{[n/n]}$ Padé approximants for the power series (5.8) by the formula*

$$(5.9) \qquad f^{[n_j/n_j]}(z) = T_{W_{2N}(z)}[g^{[n_j-n_N/n_j-n_N]}(z)], \quad j > N, \quad j \in \mathbb{N}.$$

   (ii) *the subdiagonal Padé approximants for the power series (5.1) are connected with subdiagonal $g^{[n/n-1]}$ Padé approximants for the power series (5.8) by*

$$(5.10) \qquad f^{[n_j/n_j-1]}(z) = T_{W_{2N}(z)}[g^{[n_j-n_N/n_j-n_N-1]}(z)], \quad j > N, \quad j \in \mathbb{N}.$$

Proof. Consider the induced moment problem $MP_{\kappa-\kappa_N}^{k-k_N^+}(\mathbf{s}^{(N)}, 2(n_j - n_N) - 1)$ and let

$$(5.11) \qquad W_{2(j-N)}^{(N)}(z) = \begin{pmatrix} Q_{2(j-N)-1}^{N,+}(z) & Q_{2(j-N)}^{N,+}(z) \\ P_{2(j-N)-1}^{N,+}(z) & P_{2(j-N)}^{N,+}(z) \end{pmatrix}$$

be the resolvent matrix of this moment problem. Then the matrices $W_{2j}(z)$ and $W_{2(j-N)}^{(N)}(z)$ are connected by (see (4.8))

$$(5.12) \qquad W_{2j}(z) = W_{2N}(z) W_{2(j-N)}^{(N)}(z).$$

Similarly, the resolvent matrix

$$(5.13) \qquad W_{2(j-N)-1}^{(N)}(z) = \begin{pmatrix} Q_{2(j-N)-1}^{N,+}(z) & Q_{2(j-N)-2}^{N,+}(z) \\ P_{2(j-N)-1}^{N,+}(z) & P_{2(j-N)-2}^{N,+}(z) \end{pmatrix}$$

of the moment problem $MP_{\kappa-\kappa_N}^{k-k_N}(\mathbf{s}^{(N)}, 2(n_j - n_N) - 1)$ is connected with the matrix $W_{2j-1}(z)$ by

$$(5.14) \qquad W_{2j-1}(z) = W_{2N}(z) W_{2(j-N)-1}^{(N)}(z).$$

By Proposition 5.3 diagonal Padé approximants $g^{[n_j-n_N/n_j-n_N]}(z)$ for the formal power series (5.8) are given by

$$(5.15) \qquad g^{[n_j-n_N/n_j-n_N]}(z) = \frac{Q_{2(n_j-n_N)}^{N,+}(z)}{P_{2(n_j-n_N)}^{N,+}(z)} = T_{W_{2(j-N)}^{(N)}(z)}[0], \quad j \in \mathbb{N}.$$

It follows from the factorization formula (5.12) and (5.15) that

$$T_{W_{2N}(z)}[g^{[n_j-n_N/n_j-n_N]}(z)] = T_{W_{2j}(z)}[0].$$

Hence by Proposition 5.3 $T_{W_{2N}(z)}[g^{[n_j-n_N/n_j-n_N]}(z)]$ coincides with the diagonal Padé approximants $f^{[n_j/n_j]}(z)$ for the formal power series (5.1).

By Proposition 5.4 subdiagonal Padé approximants $g^{[n_j-n_N/n_j-n_N-1]}(z)$ for the formal power series (5.8) are given by

$$(5.16) \qquad g^{[n_j-n_N/n_j-n_N-1]}(z) = \frac{Q_{2(n_j-n_N)-1}^{N,+}(z)}{P_{2(n_j-n_N)-1}^{N,+}(z)} = T_{W_{2(j-N)-1}^{(N)}(z)}[\infty], \quad j \in \mathbb{N}.$$

It follows from the factorization formula (5.14) and (5.16) that

$$T_{W_{2N}(z)}[g^{[n_j-n_N/n_j-n_N]}(z)] = T_{W_{2j}(z)}[\infty].$$

Hence by Proposition 5.4 $T_{W_{2N}(z)}[g^{[n_j-n_N/n_j-n_N-1]}(z)]$ coincides with the subdiagonal Padé approximants $f^{[n_j/n_j-1]}(z)$ for the formal power series (5.1). $\qquad\square$

Theorem 5.6. *Let $s = \{s_j\}_{j=0}^{\infty} \in \mathcal{H}_\kappa^{k,reg}$, $\kappa, k \in \mathbb{Z}_+$. Then*

(i) *If the problem $MP_\kappa^k(\mathbf{s})$ is determinate, then diagonal and subdiagonal Padé approximants converge to the unique solution of $MP_\kappa^k(\mathbf{s})$ locally uniformly on $\mathbb{C} \setminus \mathbb{R}_+$.*

(ii) *If the problem $MP_\kappa^k(\mathbf{s})$ is indeterminate, then the sequence $f^{[n/n-1]}$ of subdiagonal Padé approximants converges locally uniformly on $\mathbb{C} \setminus \mathbb{R}_+$, while the sequence $f^{[n/n]}$ of diagonal Padé approximants is not convergent but precompact in the topology of locally uniform convergence.*

Proof. Let us choose N big enough, so that

$$(5.17) \qquad \nu_-(S_j) = \kappa = \nu_-(S_{n_N}), \quad \nu_-(S_j^{(1)}) = k = \nu_-(S_{n_N}^{(1)}) \quad \text{for all} \quad j \geq n_N.$$

Then the induced sequence $\mathbf{s}^{(N)}$ belongs to the class $\mathcal{H}_0^0$ and by Lemma 5.5 the problem of convergence of diagonal and subdiagonal Padé approximants is reduced to the corresponding problem for diagonal and subdiagonal Padé approximants for the series $-\sum_{j=0}^{\infty} s_j^{(N)} z^{-(j+1)}$ corresponding to the classical Stieltjes moment problem $\mathcal{M}_0^0(\mathbf{s}^{(N)})$. The stated results for classical Stieltjes moment problems were proved in [38, Theorems 5.30, 5.31]. $\qquad\square$

6. Example. Laguerre polynomials

The monic Laguerre polynomials $\widetilde{L}_n(z,\alpha) := (-1)^n z^{-\alpha} e^z (z^{\alpha+n} e^{-z})^{(n)}$ are solutions of the three–term difference equation (see [39])

$$(6.1) \qquad y_{n+1}(z) + (2n + \alpha + 1 - z)y_n(z) + (n + \alpha)n y_{n-1}(z), \quad n \in \mathbb{N}.$$

subject to the initial conditions $\widetilde{L}_{-1}(z,\alpha) \equiv 0$, $\widetilde{L}_0(z,\alpha) \equiv 1$.

If $\alpha > -1$, then the polynomials $\{\widetilde{L}_n(z,\alpha)\}_{n=0}^{\infty}$ are orthogonal in the Hilbert space $L_2(\mathbb{R}_+, w_\alpha)$, with the weight function $w_\alpha(z) = z^\alpha e^{-z}$.

Here we consider the case when $\alpha < -1$ and α is not a negative integer. The case when α is a negative integer was treated in [35]. If $-k-1 < \alpha < -k$, $k \in \mathbb{N}$, then $\widetilde{L}_n(z,\alpha)$ are orthogonal polynomials with respect to the indefinite inner product (see [11], [37])

$$(6.2) \qquad \langle f, g \rangle_\alpha = \int_0^\infty x^\alpha \left(e^{-x} f\bar{g} - \sum_{j=0}^{k-1} (e^{-x} f\bar{g})^{(j)}(0) \frac{x^j}{j!} \right) dx.$$

Polynomials $\widetilde{Q}_n(x,\alpha)$ of the second kind are defined as solutions of (6.1) subject to the initial conditions $\widetilde{Q}_{-1}(z,\alpha) \equiv -1$ and $\widetilde{Q}_0(z,\alpha) \equiv 0$.

The P – fraction corresponding to the system (6.1) has the form (1.14) with the atoms (a_n, b_n) given by

$$(6.3) \qquad b_0 = \Gamma(1 + \alpha), \quad b_n = n(n + \alpha) \quad \text{and} \quad a_{n-1}(z) = z - 2n - \alpha + 1, \quad n \in \mathbb{N}.$$

The moments s_n for all $n \in \mathbb{N}$ are defined by

$$(6.4) \qquad s_n = \mathfrak{S}(z^n) = \Gamma(n + \alpha + 1).$$

By the Buslaev formula, see [5, formula (13)] the determinants of S_n and S_n^+ take the form

$$D_n = b_0^n b_1^{n-1} \ldots b_{n-1} = \prod_{j=1}^n (j-2)! \Gamma(\alpha+j), \quad D_n^+ = \prod_{j=1}^n (j-2)! \Gamma(\alpha+j+1), \quad n \in \mathbb{N}.$$

Therefore, the sequence $\mathbf{s} = \{s_n\}_{n=0}^\infty$ is regular and the set of its normal indices coincides with $\mathbb{N}$ and

$$\mathbf{s} \in \mathcal{H}_k^{k-1}, \quad \text{if} \quad -2k < \alpha < -2k+1, \qquad \mathbf{s} \in \mathcal{H}_k^k, \quad \text{if} \quad -2k-1 < \alpha < -2k;$$

Next, it follows from (1.6) that the parameters of the generalized S-fraction take the form

$$l_n := \frac{D_n^2}{D_n^+ D_{n-1}^+} = \frac{(n-1)! \Gamma^2(1+\alpha)}{\Gamma(1+\alpha+n)}, \quad m_n := \frac{(D_{n-1}^+)^2}{D_n D_{n-1}} = \frac{\Gamma(\alpha+n)}{(n-1)! \Gamma^2(1+\alpha)}, \quad n \in \mathbb{N}.$$

The monic Laguerre polynomials $\widetilde{L}_n(z,\alpha)$ can be calculated by (see [39], [41])

(6.5) $$\widetilde{L}_n(z,\alpha) = \sum_{k=0}^n \binom{n}{k} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} (-1)^{n+k} z^k.$$

Then using the formula

$$Q_n(z,\alpha) = \mathfrak{S}_t \left(\frac{\widetilde{L}_n(z,\alpha) - \widetilde{L}_n(t,\alpha)}{z-t} \right)$$

one can find the Lanczos polynomials of the second kind

(6.6)
$$Q_n(z,\alpha) = \mathfrak{S}_t \left(\sum_{k=0}^n \binom{n}{k} \frac{\Gamma(n+\alpha+1)}{\Gamma(k+\alpha+1)} (-1)^{n+k} \frac{z^k - t^k}{z-t} \right) =$$
$$= \sum_{k=1}^n z^{k-1} \Gamma(\alpha+n+1) \sum_{j=0}^{n-k} (-1)^{n+k+j} \binom{n}{k+j} \frac{\Gamma(\alpha+j+1)}{\Gamma(\alpha+k+j+1)}.$$

Due to (3.3) the Stieltjes polynomials of the first and second kind are calculated by

$$P_{2n}^+(z,\alpha) = \Gamma(1+\alpha) \sum_{i=0}^n \binom{n}{i} \frac{(-1)^i z^i}{\Gamma(1+\alpha+i)},$$

$$P_{2n-1}^+(z,\alpha) = -\frac{1}{(n-1)!} \prod_{j=1}^n (j+n) \left(\sum_{i=0}^{n-1} \frac{(-1)^{i-1}(n-1)! i \, z^i}{i!(n-i)! \Gamma(1+\alpha+i)} + \frac{(-1)^{n-1} z^n}{\Gamma(1+\alpha+n)} \right),$$

$$Q_{2n}^+(z,\alpha) = \sum_{i=1}^n z^{i-1} \Gamma(1+\alpha) \sum_{j=0}^{n-i} (-1)^{i+j+1} \binom{n}{i+j} \frac{\Gamma(1+\alpha+n)}{\Gamma(1+\alpha+i+j+1)},$$

$$Q_{2n-1}^+(z,\alpha) = \Gamma(1+\alpha+n) \sum_{i=1}^n z^{i-1} \sum_{j=0}^{n-i} \frac{(-1)^{i+j-1} \Gamma(1+\alpha+j)(n-1)!}{i!(n-j-i)! \Gamma(1+\alpha+i+j)}.$$

The set of solutions of the truncated moment problem $MP_\kappa^k(\mathbf{s}, 2n-1)$ is described by the formula (1.23).

Notice that the full moment problem is determinate and in the case $\alpha > -1$ its unique solution of $MP_0^0(\mathbf{s})$ is given by

$$f(z) = \int_0^\infty \frac{x^\alpha e^{-x}}{x-z} dx = z^{\frac{\alpha-1}{2}} e^{-\frac{z}{2}} W_{\frac{-\alpha-1}{2}, \frac{\alpha}{2}}(-z),$$

where $W_{\alpha,\beta}$ is the Whittaker function, see [26, 9.222].

If $-2k - 1 < \alpha < -2k$, $k \in \mathbb{N}$, then the solution $f(z)$ of $MP_k^k(\mathbf{s})$ can be found by the formula

$$(6.7) \qquad f(z) = \mathfrak{S}_x \left(\frac{e^{-x} x^\alpha}{x - z} \right) = \int_0^\infty x^\alpha \left(h(x, z) - \sum_{j=0}^{2k-1} \frac{h_x^{(j)}(0, z)}{j!} x^j \right) dx,$$

where $h(x, z) = \frac{e^{-x}}{x - z}$. Similar formula holds also for the case $-2k < \alpha < -2k + 1$. In particular, if $-2 < \alpha < -1$, then

$$f(z) = I(z) + I'(z) \in MP_1^0(\mathbf{s}), \quad \text{where} \quad I(z) = \frac{z^{\frac{\alpha}{2}} e^{-\frac{z}{2}}}{\alpha + 1} W_{\frac{-\alpha-2}{2}, \frac{\alpha+1}{2}}(-z).$$

By Propositions 5.3, 5.4 the Padé approximants of f take the form (see (5.4) and (5.6))

$$f^{[n/n]}(z) = \frac{Q_{2n}^+(z, \alpha)}{P_{2n}^+(z, \alpha)}, \quad f^{[n/n-1]}(z) = \frac{Q_{2n-1}^+(z, \alpha)}{P_{2n-1}^+(z, \alpha)}, \quad n \in \mathbb{N}.$$

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Vasyl′ Stus Donetsk National University, 21, 600-richchia str., Vinnytsia, 21021, Ukraine
E-mail address: derkach.v@gmail.com

Dragomanov National Pedagogical University, 9, Pyrogova str., Kyiv, 01601, Ukraine
E-mail address: i.m.kovalyov@gmail.com