Methods of Functional Analysis and Topology
Vol. 27 (2021), no. 4, pp. 353–369
https://doi.org/10.31392/MFAT-npu26_4.2021.09

α–REGULAR INDEFINITE STIELTJES MOMENT PROBLEM AND DARBOUX TRANSFORMATION

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Dedicated to Professor V.A. Derkach on the occasion of his 70th birthday

Abstract. A sequence of the real numbers $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$ is associated with the some indefinite Stieltjes moment problem and generalized Jacobi matrices. The relation between the α–regular indefinite Stieltjes moment problem and shifted Darboux transformation of the generalized Jacobi matrix is studied. The new formulas for the Stieltjes polynomials with the shift are found and one are used to obtain the description of the solutions of the α–regular indefinite Stieltjes moment problem.

Послідовність дійсних чисел $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$ пов'язана з деякою задачею про невизначений момент Стілтьєса та узагальненими матрицями Якобі. Досліджено зв'язок між α–регулярною проблемою невизначеного моменту Стілтьєса та зміщеним перетворенням Дарбу узагальненої матриці Якобі. Знайдено нові формули для поліномів Стілтьєса зі зсувом та використано для отримання опису розв'язків α–регулярної невизначеної проблеми моменту Стілтьєса.

1. Introduction.

A classical Stieltjes moment problem was studied in [26]. Given a sequence of the real numbers $\mathbf{s} = \{s_i\}_{i=0}^{\infty}$, find a positive measure σ with support on $\mathbb{R}_+$ such that

$$s_i = \int_{\mathbb{R}_+} t^i d\sigma(t), \qquad i \in \mathbb{Z}_+ = \mathbb{N} \cup \{0\}. \tag{1.1}$$

The problem (1.1) with a finite sequence $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$ is called a truncated Stieltjes moment problem, otherwise it is called a full Stieltjes moment problem.

The Stieltjes transform of a measure σ,

$$f(z) = \int_{\mathbb{R}_+} \frac{d\sigma(t)}{t-z}, \qquad z \in \mathbb{C}\backslash\mathbb{R}_+, \tag{1.2}$$

belongs to the Nevanlinna class $\mathbf{N}$, i.e., $f \in \mathbf{N}$, if f holomorphic on $\mathbb{C}\backslash\mathbb{R}$, $\operatorname{Im} f(z) \geq 0$, and $f(\overline{z}) = \overline{f(z)}$ for all $z \in \mathbb{C}_+$.

A function f belongs to the Stieltjes class $\mathbf{S}$, if $f \in N$ and it admits a holomorphic and nonnegative continuation to $\mathbb{R}_-$. By M.G. Krein's criterion [17],

$$f \in \mathbf{S} \Longleftrightarrow f \in \mathbf{N} \quad \text{and} \quad zf \in \mathbf{N}. \tag{1.3}$$

By the Hamburger–Nevanlinna theorem (see [1]), the truncated Stieltjes moment problem can be reformulated in terms of the transform (1.2) as an interpolation problem at ∞, namely, find $f \in \mathbf{S}$ such that the following asymptotic expansion holds:

$$f(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n}}{z^{2n+1}} + o\left(\frac{1}{z^{2n+1}}\right), \qquad z\widehat{\rightarrow}\infty. \tag{1.4}$$

2020 *Mathematics Subject Classification.* Primary 47B36 ; Secondary 47B50; 42C05; 15A23.

Keywords. moment problem, Stieltjes polynomials, Darboux transformation, m–function, monic generalized Jacobi matrix, triangular factorization.

The present research is supported by a grant of the Volkswagen Foundation.

The notation $z\widehat{\rightarrow}\infty$ means that $z \to \infty$ nontangentially, that is, inside the sector $\varepsilon < \arg z < \pi - \varepsilon$ for some $\varepsilon > 0$.

Let us recall definitions of a generalized Nevanlinna class $\mathbf{N}_\kappa$ and a generalized Stieltjes class $\mathbf{N}_\kappa^k$. A function f, meromorphic on $\mathbb{C}\backslash\mathbb{R}$ with the set of holomorphy $\mathfrak{h}_f$, is said to belong to the generalized Nevanlinna class $\mathbf{N}_\kappa$ ($\kappa \in \mathbb{N}$), if for every set $z_i \in \mathbb{C}_+ \cap \mathfrak{h}_f$ ($j = 1, \dots, n$) the form

$$\sum_{i,j=1}^{n} \frac{f(z_i) - \overline{f(z_j)}}{z_i - \overline{z}_j} \xi_i \overline{\xi}_j$$

has at most κ and for some choice of z_i ($i = 1, \dots, n$) it has exactly κ negative squares. For $f \in \mathbf{N}_\kappa$ let us write $\kappa_-(f) = \kappa$. In particular, if $\kappa = 0$, then the class $\mathbf{N}_0$ coincides with the class $\mathbf{N}$ of Nevanlinna functions. A function $f \in \mathbf{N}_\kappa$ is said to belong to the class $\mathbf{N}_\kappa^+$ (see [18, 19]) if $zf \in \mathbf{N}$ and to the class $\mathbf{N}_\kappa^k$ ($k \in \mathbb{N}$) if $zf \in \mathbf{N}_\kappa^k$ (see [3, 4, 12]). In particular, if $k = 0$, then $\mathbf{N}_\kappa^0 := \mathbf{N}_\kappa^+$. We have $f \in \mathbf{N}_\kappa^{-k}$, if $f \in \mathbf{N}_\kappa$ and $\frac{1}{z} f \in \mathbf{N}_k$ (see [11]).

Problem $\mathbf{MP}_\kappa^k(\mathbf{s}, \ell)$. Given ℓ, κ, $k \in \mathbb{Z}_+$, and a sequence of real numbers $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$, describe the set $\mathcal{M}_\kappa^k(\mathbf{s})$ of functions $f \in \mathbf{N}_\kappa^k$ that have the following asymptotic expansion:

$$f(z) = -\frac{s_0}{z^1} - \frac{s_1}{z^2} - \dots - \frac{s_\ell}{z^{\ell+1}} + o\left(\frac{1}{z^{\ell+1}}\right), \quad z\widehat{\rightarrow}\infty. \tag{1.5}$$

If $\ell = 2n - 2$ and $n \in \mathbb{N}$, then $\mathbf{MP}_\kappa^k(\mathbf{s}, \ell)$ is called an odd moment problem, otherwise $\mathbf{MP}_\kappa^k(\mathbf{s}, \ell)$ is called an even moment problem. If $\ell = \infty$, then $\mathbf{MP}_\kappa^k(\mathbf{s}, \ell)$ is called a full moment problem.

Indefinite moment problems in the classes $\mathbf{N}_\kappa$ were studied in [3, 20]. Indefinite moment problems in the classes $\mathbf{N}_\kappa^+$ and $\mathbf{N}_\kappa^k$ were studied in [20, 21] and [5, 6, 7, 8, 9, 14, 16], respectively.

In this paper, we study the $\alpha-$regular indefinite moment problem in the generalized Stieltjes class and its connection with the Darboux transformation. It is based on the results of [15, 16]. New formulas for the Stieltjes polynomials are found in the Section 3, and then we obtain a description of a solution of the $\alpha-$regular indefinite moment problem $\mathbf{MP}_\kappa^k(\mathbf{s}, \ell)$. In the sections 4-5, we investigate the Darboux transformation of the Jacobi matrices associated with the $\mathbf{MP}_\kappa^k(\mathbf{s}, \ell)$.

2. Preliminaries

In the general case (the indefinite case), the moment sequence $\mathbf{s} = \{s_i\}_{i=0}^{\infty}$ is associated with a linear functional $\mathfrak{S}$ that is defined on the linear space

$$\mathcal{P} = span\left\{z^j : j \in \mathbb{Z}_+\right\} \tag{2.6}$$

by the equality

$$\mathfrak{S}(z^j) = s_j, \qquad j \in \mathbb{Z}_+. \tag{2.7}$$

Moreover, the sequence $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$ is associated with a set of normal indices $\mathcal{N}(\mathbf{s}) = \{n_j\}_{j=1}^{N}$ defined by

$$\mathcal{N}(\mathbf{s}) = \{n_j : D_{n_j} \neq 0, j = 1, 2, \dots, N\}, \quad D_{n_j} := \det(s_{i+k})_{i,k=0}^{n_j - 1}. \tag{2.8}$$

Denote by $\nu_-(S_n)$ the number of negative eigenvalues of the Hankel matrix S_n. Let $\mathcal{H}$ be the set of finite or infinite real sequences $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$.

A sequence $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$ belongs to the class $\mathcal{H}_{\kappa,\ell}$ if

$$\nu_-(S_n) = \kappa \quad (n = [\ell/2] + 1). \tag{2.9}$$

Moreover, a sequence $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$ belongs to the class $\mathcal{H}_{\kappa,\ell}^{k}$ if $\mathbf{s} = \{s_i\}_{i=0}^{\ell} \in \mathcal{H}_{\kappa,\ell}$ and $\{s_{i+1}\}_{i=0}^{\ell-1} \in \mathcal{H}_{k,\ell-1}$, i.e., (2.9) and following condition holds:

$$\nu_-(S_n^+) = k \quad (n = [(\ell+1)/2]). \tag{2.10}$$

Using a sequence $\mathbf{s}$ (see [1, 2]) we can construct the polynomials of the first and the second kind defined by

$$P_{n_j}(z) = \frac{1}{D_{n_j}}\det\begin{pmatrix} s_0 & s_1 & \cdots & s_{n_j} \\ \cdots & \cdots & \cdots & \cdots \\ s_{n_j-1} & s_{n_j} & \cdots & s_{2n_j-1} \\ 1 & z & \cdots & z^{n_j} \end{pmatrix}, \quad Q_{n_j}(z) = \mathfrak{S}_t\left(\frac{P_{n_j}(z) - P_{n_j}(t)}{z-t}\right). \tag{2.11}$$

Furthermore, the polynomials $P_{n_i}(z)$ of the first kind and $Q_{n_i}(z)$ of the second kind are solutions of the following difference system ([25])

$$b_i y_{n_{i-1}}(z) - a_i(z) y_{n_i}(z) + y_{n_{i+1}}(z) = 0, \tag{2.12}$$

It is associated with a sequence of the atoms (a_i, b_i), $i \in \mathbb{Z}_+$ and $b_0 := s_{n_1-1}$, subject to the initial conditions

$$P_{n_{-1}}(z) \equiv 0,\ P_{n_0}(z) \equiv 1,\ Q_{n_{-1}}(z) \equiv -1,\ Q_{n_0}(z) \equiv 0. \tag{2.13}$$

A generalized Liouville-Ostrogradsky formula holds for P_{n_i} and Q_{n_i} (see [7, (2.9)]),

$$Q_{n_i}(z)P_{n_{i-1}}(z) - Q_{n_{i-1}}(z)P_{n_i}(z) = \widetilde{b}_{i-1}, \quad i \in \mathbb{N} \text{ and } \widetilde{b}_{i-1} = b_0 b_1 ... b_{i-1}. \tag{2.14}$$

The sequence $\mathbf{s}$, the system (2.12)–(2.13), the polynomials P_{n_j} and Q_{n_j} and the atoms (a_i, b_i) are associated with the following $\mathbf{P}$ – fraction (see [13, 23])

$$-\cfrac{b_0}{a_0(z) - \cfrac{b_1}{a_1(z) - \cfrac{b_2}{a_2(z) - \ddots}}} \tag{2.15}$$

and a monic generalized Jacobi matrix (see [2]),

$$\mathfrak{J} = \begin{pmatrix} \mathfrak{C}_{a_0} & \mathfrak{D}_0 & & \\ \mathfrak{B}_1 & \mathfrak{C}_{a_1} & \mathfrak{D}_1 & \\ & \mathfrak{B}_2 & \mathfrak{C}_{a_2} & \ddots \\ & & \ddots & \ddots \end{pmatrix}, \tag{2.16}$$

where the diagonal entries are companion matrices associated with the real monic polynomials a_j (see [22]),

$$\mathfrak{C}_{a_j} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ -a_0^{(j)} & -a_1^{(j)} & \cdots & -a_{\ell_j-2}^{(j)} & -a_{\ell_j-1}^{(j)} \end{pmatrix} \text{ are } \ell_j \times \ell_j \text{ matrices}, \ell_j = n_{j+1} - n_j, \tag{2.17}$$

$\mathfrak{D}_j$ and $\mathfrak{B}_{j+1}$ are $\ell_j \times \ell_{j+1}$ and $\ell_{j+1} \times \ell_j$ matrices, respectively, defined by

$$\mathfrak{D}_j = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \text{ and } \mathfrak{B}_{j+1} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 0 \\ b_{j+1} & 0 & \cdots & 0 \end{pmatrix}, b_{j+1} \in \mathbb{R}\backslash\{0\}, j \in \mathbb{Z}_+. \tag{2.18}$$

The shortened generalized Jacobi matrix $\mathfrak{J}_{[i,j]}$ is defined by

$$\mathfrak{J}_{[i,j]} = \begin{pmatrix} \mathfrak{C}_{\mathfrak{p}_i} & \mathfrak{D}_i & & \\ \mathfrak{B}_{i+1} & \mathfrak{C}_{i+1} & \ddots & \\ & \ddots & \ddots & \mathfrak{D}_{j-1} \\ & & \mathfrak{B}_j & \mathfrak{C}_{\mathfrak{p}_j} \end{pmatrix}, \; i \le j \text{ and } i,j \in \mathbb{Z}_+. \tag{2.19}$$

The following connection between the polynomials of the first and the second kind and the shortened generalized Jacobi matrices can be found in [2],

$$P_{n_j}(z) = \det(z - \mathfrak{J}_{[0,j-1]}) \quad \text{and} \quad Q_{n_j}(z) = b_0 \det(z - \mathfrak{J}_{[1,j-1]}). \tag{2.20}$$

As was shown in [2, Proposition 6.1], [10], the m-function can be found as

$$m_{[0,j-1]}(z) = -b_0 \frac{\det(z - \mathfrak{J}_{[1,j-1]})}{\det(z - \mathfrak{J}_{[0,j-1]})} = -\frac{Q_{n_j}(z)}{P_{n_j}(z)} \tag{2.21}$$

and admits the following asymptotic expansion:

$$m_{[0,j-1]}(z) = -\frac{s_0}{z} - \frac{s_1}{z^2} - \cdots - \frac{s_{2n_j-2}}{z^{2n_j-1}} + o\left(\frac{1}{z^{2n_j-1}}\right), \quad z\widehat{\rightarrow}\infty. \tag{2.22}$$

2.1. **Classification of the indefinite Stieltjes moment problems.** We set $D_n^+ := \det(s_{i+j+1})_{i,j=0}^{n-1}$. A sequence $\mathbf{s}$ is called *a regular sequence* (see [7]), if $D_{n_j}^+ \neq 0$ for all $n_j \in \mathcal{N}(\mathbf{s})$. In this case, the indefinite Stieltjes moment problem $\mathbf{MP}_\kappa^k(\mathbf{s},\ell)$ is called *a regular indefinite Stieltjes moment problem*. It was studied in [7]. Without loss of generality, the solutions of the $\mathbf{MP}_\kappa^k(\mathbf{s}, 2n_j - 2)$ can be written as

$$f(z) = \frac{1|}{|-zm_1(z)} + \frac{1|}{|l_1} + \cdots + \frac{1|}{|-m_j(z)} + \frac{1|}{|\tau(z)},$$

where the atoms (m_i, l_i) are defined by special formulas see [7], l_i are constants and the parameter τ satisfies the following conditions:

$$\tau \in \mathbf{N}_{\kappa-\kappa_N}^{k-k_N} \quad \text{and} \quad \frac{1}{\tau(z)} = o(z), \quad z\widehat{\rightarrow}\infty. \tag{2.23}$$

In the case where the sequence $\mathbf{s}$ is not regular, the indefinite Stieltjes moment problem $\mathbf{MP}_\kappa^k(\mathbf{s},\ell)$ was studied in [14]. In this case, $\mathbf{MP}_\kappa^k(\mathbf{s},\ell)$ is called *a general indefinite Stieltjes moment problem*. For example, the solutions of the odd moment problem $\mathbf{MP}_\kappa^k(\mathbf{s}, 2n_j - 2)$ take the following form:

$$f(z) = \frac{1|}{|-zm_1(z)} + \frac{1|}{|l_1(z)} + \cdots + \frac{1|}{|-m_j(z)} + \frac{1|}{|\tau(z)},$$

where the atoms (m_i, l_i) are defined by special formulas, see [14] and l_i are polynomials, the parameter τ satisfies conditions similar to (2.23).

Let us recall definitions of an $\alpha-$regular sequences.

Definition 2.1. ([16]) We say that a sequence $\mathbf{s} = \{s_i\}_{i=0}^{\ell}$ belongs to the $\alpha-$regular class $\mathcal{H}_{\kappa,\ell}^{k,\alpha-reg}$, if $\mathbf{s} \in \mathcal{H}_{\kappa,\ell}^{k}$ and all polynomials of the first kind P_{n_j} associated with the sequence $\mathbf{s}$ satisfy the following condition:

$$P_{n_j}(\alpha) \neq 0 \quad \text{for all } n_j \in \mathcal{N}(\mathbf{s}). \tag{2.24}$$

The moment problem $\mathbf{MP}_\kappa^k(\mathbf{s},\ell)$ associated with an $\alpha-$regular sequences is called *$\alpha-$regular indefinite Stieltjes moment problem.*

In this case, the solutions of the $\alpha-$regular moment problem $\mathbf{MP}_\kappa^k(\mathbf{s}, 2n_j-2)$ are

$$f(z) = \cfrac{1}{-(z-\alpha)m_1^\alpha(z) + \cfrac{1}{l_1^\alpha + \cdots + \cfrac{1}{-(z-\alpha)m_N^\alpha(z) + \cfrac{1}{\tau(z)}}}}, \tag{2.25}$$

where the atoms (m_i^α, l_i^α) are defined by (3.29), l_i^α are constants, the parameter τ satisfies conditions analogous to (2.23).

3. Stieltjes polynomials

Definition 3.1. ([16]) Let $\mathbf{s} \in \mathcal{H}_{\kappa,\ell}^{k,\alpha-reg}$. Define polynomials $P_j^+(z,\alpha)$ and $Q_j^+(z,\alpha)$ corresponding to the sequence $\mathbf{s}$ as follows:

$$\begin{aligned}
&P_{-1}^+(z,\alpha) \equiv 0, \quad P_0^+(z,\alpha) \equiv 1, \qquad Q_{-1}^+(z,\alpha) \equiv 1, \quad Q_0^+(z,\alpha) \equiv 0,\\
&P_{2i-1}^+(z,\alpha) = \frac{-1}{b_0 \dots b_{i-1}} \begin{vmatrix} P_{n_i}(z) & P_{n_{i-1}}(z) \\ P_{n_i}(\alpha) & P_{n_{i-1}}(\alpha) \end{vmatrix}, \quad P_{2i}^+(z,\alpha) = \frac{P_{n_i}(z)}{P_{n_i}(\alpha)},\\
&Q_{2i-1}^+(z,\alpha) = \frac{1}{b_0 \dots b_{i-1}} \begin{vmatrix} Q_{n_i}(z) & Q_{n_{i-1}}(z) \\ P_{n_i}(\alpha) & P_{n_{i-1}}(\alpha) \end{vmatrix}, \quad Q_{2i}^+(z,\alpha) = -\frac{Q_{n_i}(z)}{P_{n_i}(\alpha)}.
\end{aligned} \tag{3.26}$$

$P_j^+(z,\alpha)$ and $Q_j^+(z,\alpha)$ are called *the Stieltjes polynomials of the first and second kind with the shift* α, respectively.

As was shown in [16, Lemma 5.4], the Stieltjes polynomials are solutions of the following system, and $\mathbf{S}-$fraction (2.25) is also associated with the following system of difference equations:

$$\begin{cases} y_{2j} - y_{2j-2} = l_j^\alpha y_{2j-1}, \\ y_{2j+1} - y_{2j-1} = -(z-\alpha)m_{j+1}^\alpha(z) y_{2j}, \end{cases} \tag{3.27}$$

where the atoms of the $\mathbf{S}-$fraction are (m_i^α, l_i^α) and can be calculated as

$$\begin{aligned}
&d_1^\alpha = \frac{1}{b_0}, \quad l_1^\alpha = -\frac{1}{d_1^\alpha a_0(\alpha)}, \quad d_i^\alpha = \frac{1}{b_{i-1}(l_{i-1}^\alpha)^2 d_{i-1}^\alpha},\\
&m_i^\alpha(z) = \frac{(a_{i-1}(z) - a_{i-1}(\alpha))d_i^\alpha}{z-\alpha}, \quad l_i^\alpha = -\frac{l_{i-1}^\alpha}{1 + l_{i-1}^\alpha d_i^\alpha a_{i-1}(\alpha)}.
\end{aligned} \tag{3.28}$$

Moreover, it follows from (3.28) that there are relations between the atoms (m_i^α, l_i^α) and (a_i, b_i),

$$\begin{aligned}
&b_0 = \frac{1}{d_1^\alpha} \quad \text{and} \quad a_0(z) = \frac{1}{d_1^\alpha}\left((z-\alpha)m_1^\alpha(z) - \frac{1}{l_1^\alpha}\right),\\
&b_{j-1} = \frac{1}{d_{j-1}^\alpha d_j^\alpha (l_{j-1}^\alpha)^2} \quad \text{and} \quad a_{j-1}(z) = \frac{1}{d_j^\alpha}\left((z-\alpha)m_j^\alpha(z) - \left(\frac{1}{l_{j-1}^\alpha} + \frac{1}{l_j^\alpha}\right)\right).
\end{aligned} \tag{3.29}$$

Lemma 3.2. *Let $P_j^+(z,\alpha)$ and $Q_j^+(z,\alpha)$ be the Stieltjes polynomials associated with a sequence $\mathbf{s} \in \mathcal{H}_{\kappa,\ell}^{k,\alpha-reg}$. Then*

$$P_{2i}^+(z,\alpha)Q_{2i-1}^+(z,\alpha) - Q_{2i}^+(z,\alpha)P_{2i-1}^+(z,\alpha) = 1; \tag{3.30}$$

$$P_{2i-1}^+(\alpha,\alpha) = 0 \quad \textit{and} \quad P_{2i-2}^+(\alpha,\alpha) = 1; \tag{3.31}$$

$$Q_{2i-1}^+(\alpha,\alpha) = 1, \quad i \in \mathbb{N}. \tag{3.32}$$

Proof. Let us prove (3.30). By Definition 3.26, we obtain

$$\begin{aligned} P^+_{2i}(z,\alpha)Q^+_{2i-1}(z,\alpha) - Q^+_{2i}(z,\alpha)P^+_{2i-1}(z,\alpha) \\ &= \Big(P_{n_i}(z)(Q_{n_i}(z)P_{n_{i-1}}(\alpha) - Q_{n_{i-1}}(z)P_{n_i}(\alpha)) \\ &\qquad - Q_{n_i}(z)(P_{n_i}(z)P_{n_{i-1}}(\alpha) - P_{n_{i-1}}(z)P_{n_i}(\alpha))\Big) \Big/ \Big(\widetilde{b}_{i-1}P_{n_i}(\alpha)\Big) \\ &= \frac{Q_{n_i}(z)P_{n_{i-1}}(z) - P_{n_i}(z)Q_{n_{i-1}}(z)}{\widetilde{b}_{i-1}} = \{\text{by } (2.14)\} = 1. \end{aligned}$$

Formula (3.31) immediately follows from (3.26). Calculating $Q^+_{2i-1}(\alpha,\alpha)$, we get

$$Q^+_{2i-1}(\alpha,\alpha) = \frac{Q_{n_i}(\alpha)P_{n_{i-1}}(\alpha) - P_{n_i}(\alpha)Q_{n_{i-1}}(\alpha)}{\widetilde{b}_{i-1}} = 1.$$

This completes the proof. □

Lemma 3.3. *Let a sequence of polynomials $\{y_{n_i}(z)\}_{i=0}^{\infty}$ satisfy the three-term recurrence relation (2.12) and let deg $(y_{n_i}) = n_i - n_{i-1} \geq 1$. Then the generalized Christoffel–Darboux formula holds,*

$$y_{n_{N+1}}(z)y_{n_N}(x) - y_{n_N}(z)y_{n_{N+1}}(x) = \sum_{i=0}^{N} \frac{\widetilde{b}_N}{\widetilde{b}_{N-i}}(a_i(z) - a_i(x))y_{n_i}(z)y_{n_i}(x). \tag{3.33}$$

Proof. Let us find a recurrence formula. Let $j \in \mathbb{Z}_+$, then

$$\begin{aligned} y_{n_{j+1}}(z)y_{n_j}(x) - y_{n_j}(z)y_{n_{j+1}}(x) = \{\text{by } (2.12)\} \\ &= (a_j(z)y_{n_j}(z) - b_j y_{n_{j-1}}(z))y_{n_j}(x) - (a_j(x)y_{n_j}(x) - b_j y_{n_{j-1}}(x))y_{n_j}(z) \\ &= (a_j(z) - a_j(x))y_{n_j}(z)y_{n_j}(x) + b_j(y_{n_j}(z)y_{n_{j-1}}(x) - y_{n_j}(x)y_{n_{j-1}}(z)). \end{aligned} \tag{3.34}$$

Due to $\widetilde{b}_j = b_0 b_1 \dots b_{j-1}$ and applying the recurrence formula (3.34) $N-$times to

$$y_{n_{N+1}}(z)y_{n_N}(x) - y_{n_N}(z)y_{n_{N+1}}(x),$$

the generalized Christoffel–Darboux formula (3.33) is obtained. This completes the proof. □

Remark 3.4. If $n_{j+1} - n_j = 1$ for all $j \in \mathbb{Z}_+$. Then (3.33) is the classical Christoffel–Darboux formula, i.e.,

$$\frac{y_{n_{N+1}}(z)y_{n_N}(x) - y_{n_N}(z)y_{n_{N+1}}(x)}{z - x} = \sum_{i=0}^{N} \frac{\widetilde{b}_N}{\widetilde{b}_{N-i}} y_{n_i}(z)y_{n_i}(x) \tag{3.35}$$

Proof. Due to $n_{j+1} - n_j = 1$ for all $j \in \mathbb{Z}_+$, we have $a_j(z) - a_j(x) = z - x$ and then

$$y_{n_{N+1}}(z)y_{n_N}(x) - y_{n_N}(z)y_{n_{N+1}}(x) = \sum_{i=0}^{N} \frac{\widetilde{b}_N}{\widetilde{b}_{N-i}}(z - x)y_i(z)y_i(x)$$

. Consequently, we obtain (3.35). This completes the proof. □

Theorem 3.5. *Let $\{P_{n_j}\}_{j=0}^{\infty}$ and $\{Q_{n_j}\}_{j=0}^{\infty}$ be sequences of the polynomials of the first and the second kind, respectively, associated with the three-term recurrence relation (2.12) and let $\alpha \in \mathbb{R}$ be such that $P_{n_j}(\alpha) \neq 0$ for all $j \in \mathbb{Z}_+$. Then the generalized Christoffel–Darboux formula for P_{n_j} takes the following form:*

$$\frac{P_{n_{N+1}}(z)P_{n_N}(\alpha) - P_{n_N}(z)P_{n_{N+1}}(\alpha)}{z - \alpha} = \sum_{i=0}^{N} \frac{\widetilde{b}_N}{\widetilde{b}_{N-i}} \frac{m^{\alpha}_{i+1}(z)}{d^{\alpha}_{i+1}} P_{n_i}(z)P_{n_i}(\alpha), \tag{3.36}$$

where the polynomials m^{α}_i and the numbers d^{α}_i are defined by (3.28).

Furthermore, the polynomials of the second kind, Q_{n_j}, satisfy the following:

$$\frac{Q_{n_{N+1}}(z)P_{n_N}(\alpha)-Q_{n_N}(z)P_{n_{N+1}}(\alpha)}{z-\alpha}=\sum_{i=0}^{N}\frac{\widetilde{b}_N}{\widetilde{b}_{N-i}}\frac{m_{i+1}^{\alpha}(z)}{d_{i+1}^{\alpha}}Q_{n_i}(z)P_{n_i}(\alpha). \tag{3.37}$$

Proof. Since the polynomials P_{n_j} satisfy (2.12) and it follows from (3.33) that

$$\begin{aligned}\frac{P_{n_{N+1}}(z)P_{n_N}(\alpha)-P_{n_N}(z)P_{n_{N+1}}(\alpha)}{z-\alpha}&=\frac{1}{z-\alpha}\sum_{i=0}^{N}\frac{\widetilde{b}_N}{\widetilde{b}_{N-i}}(a_i(z)-a_i(\alpha))P_{n_i}(z)P_{n_i}(\alpha)\\&=\{\text{by (3.28)}\}=\sum_{i=0}^{N}\frac{\widetilde{b}_N}{\widetilde{b}_{N-i}}\frac{m_{i+1}^{\alpha}(z)}{d_{i+1}^{\alpha}}Q_{n_i}(z)P_{n_i}(\alpha).\end{aligned}$$

Let us find a recurrence formula for Q_{n_j} similar to (3.34),

$$\begin{aligned}Q_{n_{j+1}}(z)P_{n_j}(\alpha)&-Q_{n_j}(z)P_{n_{j+1}}(\alpha)=\{\text{by (2.12)}\}\\&=(a_j(z)Q_{n_j}(z)-b_jQ_{n_{j-1}}(z))P_{n_j}(\alpha)-(a_j(\alpha)P_{n_j}(\alpha)-b_jP_{n_{j-1}}(\alpha))Q_{n_j}(z)\\&=(a_j(z)-a_j(\alpha))Q_{n_j}(z)P_{n_j}(\alpha)+b_j(Q_{n_j}(z)P_{n_{j-1}}(\alpha)-P_{n_j}(\alpha)Q_{n_{j-1}}(z)).\end{aligned} \tag{3.38}$$

Applying (3.38) $N-$times, we get

$$\begin{aligned}\frac{Q_{n_{N+1}}(z)P_{n_N}(\alpha)-Q_{n_N}(z)P_{n_{N+1}}(\alpha)}{z-\alpha}&=\frac{1}{z-\alpha}\sum_{i=0}^{N}\frac{\widetilde{b}_N}{\widetilde{b}_{N-i}}(a_i(z)-a_i(\alpha))P_{n_i}(z)P_{n_i}(\alpha)\\&=\sum_{i=0}^{N}\frac{\widetilde{b}_N}{\widetilde{b}_{N-i}}\frac{m_{i+1}^{\alpha}(z)}{d_{i+1}^{\alpha}}Q_{n_i}(z)P_{n_i}(\alpha).\end{aligned}$$

So, (3.36) and (3.37) are proved. This completes the proof. □

Hence, we can rewrite representation of the Stieltjes polynomilas in terms of the Christoffel–Darboux formula.

Corollary 3.6. *Let $P_i^+(z,\alpha)$ and $Q_i^+(z,\alpha)$ be Stieltjes polynomials of the first and the second kind with a shift α, respectively. Then the Stieltjes polynomials of the first and the second kind can be found as*

$$P_{2N-1}^+(z,\alpha)=-(z-\alpha)\sum_{i=0}^{N-1}\frac{m_{i+1}^{\alpha}(z)}{\widetilde{b}_{N-1-i}d_{i+1}^{\alpha}}P_{n_i}(z)P_{n_i}(\alpha); \tag{3.39}$$

$$Q_{2N-1}^+(z,\alpha)=(z-\alpha)\sum_{i=0}^{N-1}\frac{m_{i+1}^{\alpha}(z)}{\widetilde{b}_{N-1-i}d_{i+1}^{\alpha}}Q_{n_i}(z)P_{n_i}(\alpha). \tag{3.40}$$

Theorem 3.7. *Let $\mathbf{s}=\{s_i\}_{i=0}^{2n_N-2}\in\mathcal{H}_{\kappa',\ell}^{k',\alpha-reg}$, and let $\mathcal{N}(\boldsymbol{s})=\{n_j\}_{j=1}^{N}$. Let $\alpha\in\mathbb{R}$ be such that $P_{n_j}(\alpha)\neq 0$ for all $j=\overline{1,N}$, and let the generalized Stieltjes polynomials $P_j^+(z,\alpha)$ and $Q_j^+(z,\alpha)$ be defined by (3.26). Then we have the following:*

(1) *$f\in\mathcal{M}_\kappa^k(\boldsymbol{s},2n_N-2)$ if and only if f admits the representation*

$$f(z)=\frac{(z-\alpha)P_{n_{N-1}}(\alpha)\sum\limits_{i=0}^{N-1}\frac{m_{i+1}^{\alpha}(z)}{\widetilde{b}_{N-1-i}d_{i+1}^{\alpha}}Q_{n_i}(z)P_{n_i}(\alpha)\tau(z)-Q_{n_{N-1}}(z)}{(\alpha-z)P_{n_{N-1}}(\alpha)\sum\limits_{i=0}^{N-1}\frac{m_{i+1}^{\alpha}(z)}{\widetilde{b}_{N-1-i}d_{i+1}^{\alpha}}P_{n_i}(z)P_{n_i}(\alpha)\tau(z)+P_{n_{N-1}}(z)}, \tag{3.41}$$

where the parameter τ satisfies the following conditions:

$$\tau\in N_{\kappa_N}^{k_N}\quad\text{and}\quad\frac{1}{\tau(z)}=o(z),\quad z\widehat{\to}\infty, \tag{3.42}$$

the indices κ_N and k_N are calculated by the following formulas:

$$\begin{aligned} \kappa_N &= \kappa - \sum_{j=1}^{N} \kappa_-((z-\alpha)m_j^\alpha) \quad \text{and} \quad k_N \geq k - \widetilde{k}_N, \\ \widetilde{k}_N &= \sum_{j=1}^{N} \kappa_-(m_j^\alpha) + \sum_{j=1}^{N-1} \kappa_-((z-\alpha)l_j^\alpha). \end{aligned} \tag{3.43}$$

(2) $f \in \mathcal{M}_\kappa^k(\boldsymbol{s}, 2n_N - 1)$ *if and only if* f *admits the representation*

$$f(z) = \frac{(z-\alpha)P_{n_{N-1}}(\alpha) \sum\limits_{i=0}^{N-1} \frac{m_{i+1}^\alpha(z)}{\widetilde{b}_{N-1-i} d_{i+1}^\alpha} Q_{n_i}(z) P_{n_i}(\alpha)\tau(z) - Q_{n_N}(z)}{(\alpha - z)P_{n_{N-1}}(\alpha) \sum\limits_{i=0}^{N-1} \frac{m_{i+1}^\alpha(z)}{\widetilde{b}_{N-1-i} d_{i+1}^\alpha} P_{n_i}(z) P_{n_i}(\alpha)\tau(z) + P_{n_N}(z)}, \tag{3.44}$$

where the parameter τ satisfies the following conditions:

$$\tau \in N_{\kappa_N}^{k_N^+} \quad \text{and} \quad \tau(z) = o(1), \quad z \widehat{\to} \infty, \tag{3.45}$$

the indices κ_N and k_N are calculated by the following formulas:

$$\begin{aligned} \kappa_N &= \kappa - \sum_{j=1}^{N} \kappa_-((z-\alpha)m_j^\alpha) \quad \text{and} \quad k_N^+ \geq k - \widetilde{k}_N^+, \\ \widetilde{k}_N^+ &= \sum_{j=1}^{N} \kappa_-(m_j^\alpha) + \sum_{j=1}^{N} \kappa_-((z-\alpha)l_j^\alpha). \end{aligned} \tag{3.46}$$

Lemma 3.8. *Let $\mathbf{s} \in \mathcal{H}_{\kappa,\ell}^{k,\alpha-reg}$ and let the $\mathbf{S}$−fraction (2.25) be associated with the sequence $\mathbf{s}$. Then we have the following:*

(1) *the lengths l_i^α can be calculated by*

$$l_i^\alpha = Q_{2i}^+(\alpha,\alpha) - Q_{2i-2}^+(\alpha,\alpha) \quad \text{and} \quad l_i^\alpha = -\frac{Q_{n_i}(\alpha)}{P_{n_i}(\alpha)} + \frac{Q_{n_{i-1}}(\alpha)}{P_{n_{i-1}}(\alpha)}. \tag{3.47}$$

(2) *For all $N \in \mathbf{N}$ the following formulas hold:*

$$\sum_{i=1}^{N} l_i^\alpha = Q_{2N}^+(\alpha,\alpha) \quad \text{and} \quad \sum_{i=1}^{N} l_i^\alpha = -\frac{Q_{n_N}(\alpha)}{P_{n_N}(\alpha)}; \tag{3.48}$$

$$\sum_{i=1}^{N} m_i^\alpha(\alpha) = -P_{2N-1}^{+'}(\alpha,\alpha). \tag{3.49}$$

Proof. Due to (3.27), we obtain

$$Q_{2i}^+(\alpha,\alpha) - Q_{2i-2}^+(\alpha,\alpha) = l_i^\alpha Q_{2i-1}^+(\alpha,\alpha).$$

By Lemma 3.2 (see (3.32)) and by (3.26), we get

$$l_i^\alpha = Q_{2i}^+(\alpha,\alpha) - Q_{2i-2}^+(\alpha,\alpha) = -\frac{Q_{n_i}(\alpha)}{P_{n_i}(\alpha)} + \frac{Q_{n_{i-1}}(\alpha)}{P_{n_{i-1}}(\alpha)}.$$

Consequently,

$$\sum_{i=1}^{N} l_i^\alpha = -\frac{Q_{n_1}(\alpha)}{P_{n_1}(\alpha)} + \frac{Q_{n_0}(\alpha)}{P_{n_0}(\alpha)} - \cdots - \frac{Q_{n_N}(\alpha)}{P_{n_N}(\alpha)} + \frac{Q_{n_{N-1}}(\alpha)}{P_{n_{N-1}}(\alpha)} = -\frac{Q_{n_N}(\alpha)}{P_{n_N}(\alpha)} = Q_{2N}^+(\alpha,\alpha).$$

By (3.27) we obtain

$$P_{2i+1}^{+'}(z,\alpha) - P_{2i-1}^{+'}(z,\alpha) = -(z-\alpha)(m_{i+1}^\alpha(z) P_{2i}^+(z,\alpha))' - m_{i+1}^\alpha(z) P_{2i}^+(z,\alpha).$$

Hence

$$m_{i+1}^{\alpha}(z) = \frac{P_{2i-1}^{+'}(z,\alpha) - P_{2i+1}^{+'}(z,\alpha)}{P_{2i}^{+}(z,\alpha)}|_{z=\alpha} = P_{2i-1}^{+'}(\alpha,\alpha) - P_{2i+1}^{+'}(\alpha,\alpha)$$

and we get

$$\begin{aligned}\sum_{i=1}^{N} m_i^{\alpha}(\alpha) &= P_{-1}^{+'}(\alpha,\alpha) - P_{1}^{+'}(\alpha,\alpha) + \cdots + P_{2N-3}^{+'}(\alpha,\alpha) - P_{2N-1}^{+'}(\alpha,\alpha) \\ &= -P_{2N-1}^{+'}(\alpha,\alpha).\end{aligned}$$

This completes the proof. □

Lemma 3.9. *Let* $\mathbf{s} \in \mathcal{H}_{\kappa',\ell}^{k',\alpha-reg}$*, and let* $\mathbf{s}$ *be associated with the* $\mathbf{S}-$*fraction* (2.25)*. Then the atoms* $(m_i^{\alpha}, l_i^{\alpha})$ *satisfy the following:*

(1) *the leading coefficient of the polynomial* m_i^{α} *is calculated as*

$$d_i^{\alpha} = \frac{P_{n_{i-1}}^2(\alpha)}{\widetilde{b}_{i-1}}, \quad \text{where}\, \widetilde{b}_{i-1} = b_0 b_1 \dots b_{i-1}; \tag{3.50}$$

(2) *the numbers (lengths)* l_i^{α} *can be found by*

$$l_i^{\alpha} = -\frac{\widetilde{b}_{i-1}}{P_{n_{i-1}}(\alpha) P_{n_i}(\alpha)}; \tag{3.51}$$

(3) d_i^{α} *and* l_i^{α} *are connected by the following:*

$$d_i^{\alpha} l_i^{\alpha} = -\frac{P_{n_{i-1}}(\alpha)}{P_{n_i}(\alpha)}, \tag{3.52}$$

(4) d_{i+1}^{α} *and* l_i^{α} *are connected by the following:*

$$d_{i+1}^{\alpha} l_i^{\alpha} = -\frac{P_{n_i}(\alpha)}{b_i P_{n_{i-1}}(\alpha)}, \quad \text{for all } i \in \mathbb{N}. \tag{3.53}$$

Proof. By (2.12), (2.13), and (3.29), we obtain

$$\begin{aligned} &P_{n_1}(z) = a_0(z), \quad d_1^{\alpha} = \frac{1}{b_0}, \quad d_{i+1}^{\alpha} = \frac{1}{(l_i^{\alpha})^2 d_i^{\alpha} b_i}, \\ &a_0(\alpha) = -\frac{1}{d_1^{\alpha} l_i^{\alpha}}, \quad a_i(\alpha) = -\frac{1}{d_{i+1}^{\alpha} l_i^{\alpha}} - \frac{1}{d_{i+1}^{\alpha} l_{i+1}^{\alpha}}. \end{aligned} \tag{3.54}$$

Hence, by induction, we get
Base case: for $i = 1$,

$$d_1^{\alpha} = \frac{1}{b_0} = \{\text{by}(2.13)\} = \frac{P_{n_0}^2(\alpha)}{b_0},$$

$$-\frac{1}{d_1^{\alpha} l_1^{\alpha}} = P_{n_1}(\alpha) \Rightarrow l_1^{\alpha} = -\frac{1}{d_1^{\alpha} P_{n_1}(\alpha)} = -\frac{b_0}{P_{n_1}(\alpha)} = -\frac{b_0}{P_{n_0}(\alpha) P_{n_1}(\alpha)}.$$

Induction step: let (3.50) and (3.51) hold for $i = N-1$. Due to (3.51) and (3.54), we obtain

$$\begin{aligned} d_N^{\alpha} &= \frac{1}{(l_{N-1}^{\alpha})^2 d_{N-1}^{\alpha} b_{N-1}} = \frac{1}{(l_{N-1}^{\alpha})^2 d_{N-1}^{\alpha} b_{N-1}} = \frac{P_{n_{N-2}}^2(\alpha) P_{n_{N-1}}^2(\alpha) \widetilde{b}_{N-2}}{\widetilde{b}_{N-2}^2 P_{n_{N-2}}^2(\alpha) b_{N-1}} \\ &= \frac{P_{n_{N-1}}^2(\alpha)}{\widetilde{b}_{N-1}}. \end{aligned}$$

Substituting d_N into (2.12) we get l_N as

$$b_{N-1}P_{n_{N-2}}(\alpha) - a_{N-1}(\alpha)P_{n_{N-1}}(\alpha) + P_{n_N}(\alpha) = 0,$$

$$b_{N-1}P_{n_{N-2}}(\alpha) + \left(\frac{1}{d_N^\alpha l_{N-1}^\alpha} + \frac{1}{d_N^\alpha l_N^\alpha}\right) P_{n_{N-1}}(\alpha) + P_{n_N}(\alpha) = 0,$$

$$b_{N-1}P_{n_{N-2}}(\alpha) - \frac{\widetilde{b}_{N-1}P_{n_{N-2}}(\alpha)P_{n_{N-1}}(\alpha)}{P^2_{n_{N-1}}(\alpha)\widetilde{b}_{N-2}} P_{n_{N-1}}(\alpha) + \frac{\widetilde{b}_{N-1}}{P^2_{n_{N-1}}(\alpha)l_N^\alpha} P_{n_{N-1}}(\alpha) + P_{n_N}(\alpha) = 0.$$

Consequently, we obtain

$$\frac{\widetilde{b}_{N-1}}{P^2_{n_{N-1}}(\alpha)l_N^\alpha} P_{n_{N-1}}(\alpha) + P_{n_N}(\alpha) = 0 \Rightarrow l_N^\alpha = -\frac{\widetilde{b}_{n_{N-1}}}{P_{n_{N-1}}(\alpha)P_{n_N}(\alpha)}.$$

So, (3.50) and (3.51) are proved. Let us show the validity of the formulas (3.52) and (3.53):

$$d_i^\alpha l_i^\alpha = -\frac{P^2_{i-1}(\alpha)}{\widetilde{b}_{i-1}} \frac{\widetilde{b}_{i-1}}{P_{i-1}(\alpha)P_i(\alpha)} = -\frac{P_{i-1}(\alpha)}{P_i(\alpha)},$$

$$d_{i+1}^\alpha l_i^\alpha = -\frac{P^2_{n_i}(\alpha)}{\widetilde{b}_i} \frac{\widetilde{b}_{i-1}}{P_{n_{i-1}}(\alpha)P_{n_i}(\alpha)} = -\frac{P_{n_i}(\alpha)}{b_i P_{n_{i-1}}(\alpha)}.$$

This completes the proof. □

Corollary 3.10. *Let* $\mathbf{s} \in \mathcal{H}_{\kappa',\ell}^{k',\alpha-reg}$*, and let* $\mathbf{s}$ *be associated with the* $\mathbf{S}-$*fraction* (2.25)*. Then polynomials of the first kind,* P_{n_j}*, for* α *are calculated by*

$$P_{n_j}(\alpha) = (-1)^j \prod_{i=1}^{j} \frac{1}{d_i^\alpha l_i^\alpha}. \tag{3.55}$$

Moreover, the following alternative formula holds:

$$P_{n_j}(\alpha) = (-1)^j \frac{\widetilde{b}_j}{b_0} \prod_{i=1}^{j} d_{i+1}^\alpha l_i^\alpha. \tag{3.56}$$

4. Shifted Darboux transformation of the monic Jacobi matrices

Now we consider the Darboux transformation with a shift α of a monic Jacobi matrix associated a sequence $\mathbf{s} = \{s_i\}_{i=0}^\infty \in \mathcal{H}_{\kappa'}^{k'}$ such that $n_{j+1} - n_j = 1$ for all $j \in \mathbb{Z}_+$. In this case, the monic Jacobi matrix J takes the following form:

$$J = \begin{pmatrix} a_0 & 1 & \\ b_1 & a_1 & \ddots \\ & \ddots & \ddots \end{pmatrix}. \tag{4.57}$$

The three-term difference relation (2.12) for the matrix J can be rewritten as

$$\begin{aligned} zy_0 &= \frac{1}{d_1}y_{-1} + \left(\alpha + \frac{1}{d_1 l_1^\alpha}\right) y_0 + y_1, \\ zy_j &= \frac{1}{d_1 d_{j+1}(l_j^\alpha)^2} y_{j-1} + \left(\alpha + \frac{1}{d_{j+1}}\left(\frac{1}{l_{j+1}^\alpha} + \frac{1}{l_j^\alpha}\right)\right) y_i + y_{j+1}, \end{aligned} \tag{4.58}$$

subject to the initial conditions

$$P_{-1}(z) \equiv 0, \quad P_0(z) \equiv 1, \quad \text{and} \quad Q_{-1}(z) \equiv -1, \quad Q_0(z) \equiv 0. \tag{4.59}$$

By [15] we find $\mathfrak{LU}-$ factorization with the shift α of the monic Jacobi matrix J.

Proposition 4.1. *Let J be a monic Jacobi matrix associated with a sequence $\mathbf{s} = \{s_i\}_{i=0}^{\infty} \in \mathcal{H}_{\kappa'}^{k',\alpha-reg}$, and let $\mathcal{N}(\mathbf{s}) = \{n_j\}_{j=0}^{\infty}$ be a set of normal indices of the sequence $\mathbf{s}$, where $n_j = j$ and $n_0 = 0$. Let P_j be polynomials of the first kind corresponding to the matrix J, and let α be some real number such that $P_j(\alpha) \neq 0$ for all $j \in \mathbb{Z}_+$. Then the monic Jacobi matrix J admits the following $\mathfrak{L}\mathfrak{U}$ – factorization with the shift α:*

$$J = \mathfrak{L}\mathfrak{U} + \alpha I, \tag{4.60}$$

where the factorization matrices $\mathfrak{L}$ and $\mathfrak{U}$ are defined by

$$\mathfrak{L} = \begin{pmatrix} 1 & & & \\ \mathfrak{l}_1 & 1 & & \\ & \mathfrak{l}_2 & 1 & \\ & & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad \mathfrak{U} = \begin{pmatrix} -\mathfrak{u}_0 & 1 & & \\ & -\mathfrak{u}_1 & 1 & \\ & & -\mathfrak{u}_2 & \ddots \\ & & & \ddots \end{pmatrix}, \tag{4.61}$$

the entries of the matrices $\mathfrak{L}$ and $\mathfrak{U}$ are calculated by

$$\mathfrak{l}_j = \frac{1}{d_{j+1}^{\alpha} l_j^{\alpha}} \quad \text{and} \quad \mathfrak{u}_{j-1} = -\frac{1}{d_j^{\alpha} l_j^{\alpha}}, \quad \text{for all } j \in \mathbb{N}. \tag{4.62}$$

Furthermore,

$$P_j(\alpha) = \prod_{i=0}^{j-1} \mathfrak{u}_i, \quad \text{for all } j \in \mathbb{N}. \tag{4.63}$$

Proof. By [15, Lemma 3.2], we get $\mathfrak{L}$ and $\mathfrak{U}$ matrices, which are defined by (4.62). In this case, the matrix J takes the following form:

$$J = \mathfrak{L}\mathfrak{U} + \alpha = \begin{pmatrix} \alpha + \frac{1}{d_1^{\alpha} l_1^{\alpha}} & 1 & & \\ \frac{1}{d_1^{\alpha} d_2^{\alpha} (l_1^{\alpha})^2} & \alpha + \frac{1}{d_2^{\alpha}}\left(\frac{1}{l_1^{\alpha}} + \frac{1}{l_2^{\alpha}}\right) & 1 & \\ & \frac{1}{d_2^{\alpha} d_3^{\alpha} (l_2^{\alpha})^2} & \alpha + \frac{1}{d_3^{\alpha}}\left(\frac{1}{l_2^{\alpha}} + \frac{1}{l_3^{\alpha}}\right) & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{4.64}$$

Moreover, by [15, Lemma 3.3],

$$P_j(\alpha) = \prod_{i=0}^{j-1} \mathfrak{u}_k = (-1)^j \prod_{i=0}^{j-1} \left(\frac{1}{d_i^{\alpha} l_i^{\alpha}}\right).$$

This completes the proof. □

Proposition 4.2. *Let $\mathbf{s} = \{s_i\}_{i=0}^{\infty} \in \mathcal{H}_{\kappa'}^{k',\alpha-reg}$ be associated with a monic Jacobi matrix J, and let $\mathcal{N}(\mathbf{s}) = \{n_j\}_{j=0}^{\infty}$ be a set of normal indices of the sequence $\mathbf{s}$, where $n_j = j$ and $n_0 = 0$. Let $J = \mathfrak{L}\mathfrak{U} + \alpha I$ be its $\mathfrak{L}\mathfrak{U}$ – factorization with the shift α of the form (4.60)–(4.62). Then we have the following:*

(1) *the shifted Darboux transformation without parameter of J is a monic Jacobi matrix $J^{(p)} = \mathfrak{U}\mathfrak{L} + \alpha$ such that*

$$J^{(p)} = \begin{pmatrix} \alpha + \frac{1}{l_1^{\alpha}}\left(\frac{1}{d_1^{\alpha}} + \frac{1}{d_2^{\alpha}}\right) & 1 & & \\ \frac{1}{(d_2^{\alpha})^2 l_1^{\alpha} l_2^{\alpha}} & \alpha + \frac{1}{l_2^{\alpha}}\left(\frac{1}{d_2^{\alpha}} + \frac{1}{d_3^{\alpha}}\right) & 1 & \\ & \frac{1}{(d_3^{\alpha})^2 l_2^{\alpha} l_3^{\alpha}} & \alpha + \frac{1}{l_3^{\alpha}}\left(\frac{1}{d_3^{\alpha}} + \frac{1}{d_4^{\alpha}}\right) & \ddots \\ & & \ddots & \ddots \end{pmatrix}. \tag{4.65}$$

(2) *The polynomials of the first and the second kind of the matrix $J^{(p)}$ take the following form:*

$$P_{j-1}^{(p)}(z)=-\frac{P_{j-1}(\alpha)}{d_1^\alpha(z-\alpha)}P_{2j-1}^{+}(z,\alpha) \text{ and } Q_{j-1}^{(p)}(z)=\frac{P_{j-1}(\alpha)}{d_1^\alpha}Q_{2j-1}^{+}(z,\alpha),\ j\in\mathbb{N}. \tag{4.66}$$

(3) *The m – function of the monic Jacobi matrix $J^{(p)}$ is*

$$m_{[0,j-1]}^{(p)}(z)=\frac{(z-\alpha)Q_{2j-1}^{+}(z,\alpha)}{P_{2j-1}^{+}(z,\alpha)}. \tag{4.67}$$

Furthermore, $m_{[0,j-1]}^{(p)}$ admits the following asymptotic expansion:

$$m_{[0,j-1]}^{(p)}(z)=-\frac{s_1-\alpha s_0}{z}-\cdots-\frac{s_{2j-1}-\alpha s_{2j-2}}{z^{2j-1}}+o\left(\frac{1}{z^{2j-1}}\right),\quad z\widehat{\rightarrow}\infty. \tag{4.68}$$

Proof. Calculating $\mathfrak{UL}+\alpha I$ we obtain (4.65). By [15, Theorwm 3.10],

$$P_{j-1}^{(p)}(z)=\frac{1}{z-\alpha}\left(P_j(z)-\frac{P_j(\alpha)}{P_{j-1}(\alpha)}P_{j-1}(z)\right)=\frac{P_j(z)P_{j-1}(\alpha)-P_j(\alpha)P_{j-1}(z)}{(z-\alpha)P_{j-1}(\alpha)}.$$

On the other hand, by (3.29), (3.26), and (4.63),

$$P_{2i-1}^{+}(z,\alpha)=-\frac{P_j(z)P_{j-1}(\alpha)-P_j(\alpha)P_{j-1}(z)}{\frac{1}{d_1^\alpha}P_{j-1}^2(\alpha)}.$$

Consequently, $P_{j-1}^{(p)}(z)=-\frac{P_{j-1}(\alpha)}{d_1^\alpha(z-\alpha)}P_{2j-1}^{+}(z,\alpha)$ for all $j\in\mathbb{N}$.

By [15, Theorwm 3.13],

$$Q_{j-1}^{(p)}(z)=\left(Q_j(z)-\frac{P_j(\alpha)}{P_{j-1}(\alpha)}Q_{j-1}(z)\right)=\frac{Q_j(z)P_{j-1}(\alpha)-P_j(\alpha)Q_{j-1}(z)}{P_{j-1}(\alpha)}$$

and by (3.29), (3.26), (4.63),

$$Q_{2j-1}^{+}(z,\alpha)=\frac{Q_j(z)P_{j-1}(\alpha)-P_j(\alpha)Q_{j-1}(z)}{\frac{1}{d_1^\alpha}P_{j-1}^2(\alpha)}.$$

Therefore, we get $Q_{j-1}^{(p)}(z)=\frac{P_{j-1}(\alpha)}{d_1^\alpha}Q_{2j-1}^{+}(z,\alpha)$ for all $j\in\mathbb{N}$.

Statement (3) directly follows from (2.21) and (4.66). This competes the proof. □

Corollary 4.3. *Let $J=\mathfrak{LU}+\alpha I$ be its $\mathfrak{LU}$ – factorization with a shift α of the form (4.60)–(4.62) and let $J^{(p)}=\mathfrak{UL}+\alpha I$ be the shifted Darboux transformation of the matrix J. Let $P_i^{+}(z,\alpha)$ and $Q_i^{+}(z,\alpha)$ be Stieltjes polynomials of the first and the second kind with the shift α, respectively. Then the m – function of the matrix J can be calculated by the following formula:*

$$m_{[0,j-1]}(z)=\frac{Q_{2j-1}^{+}(z,\alpha)}{P_{2j-1}^{+}(z,\alpha)}-\frac{s_0}{z-\alpha}. \tag{4.69}$$

Proof. By [15, Proposition 3.19],

$$m_{[0,j-1]}^{(p)}(z)=(z-\alpha)m_{[0,j-1]}(z)+s_0$$

and by (4.67), we obtain (4.69). □

Remark 4.4. According to formula (4.65), we can see that the "lengths" l_j^α and the "masses" d_j^α switch places. If we set $d_{j+1}:=l_j^\alpha$ and $l_j:=d_j^\alpha$ for all $j\in\mathbb{N}$, then we obtain that $J^{(p)}$ is associated with some part of the Krein-Stieltjes string with the atoms (d_i,l_i) (i.e., in our case, we mean that $J^{(p)}$ is associated with the full Krein-Stieltjes string without the first atom (d_1,l_1)).

Proposition 4.5. *Let* $\mathbf{s}=\{s_i\}_{i=0}^{\infty}\in\mathcal{H}_{\kappa'}^{k',\alpha-reg}$ *be associated with a monic Jacobi matrix* J*, and let* $\mathcal{N}(\mathbf{s})=\{n_j\}_{j=0}^{\infty}$ *be a set of normal indices of the sequence* $\mathbf{s}$*, where* $n_j=j$ *and* $n_0=0$*. Let* $J=\mathfrak{L}\mathfrak{U}+\alpha I$ *be its* $\mathfrak{L}\mathfrak{U}$ *– factorization with a shift* α *of the form* (4.60)–(4.62) *and* $J^{(p)}=\mathfrak{U}\mathfrak{L}+\alpha I$ *be its shifted Darboux transformation without parameter. Then* $J^{(p)}$ *is associated with the following* $J-$*fraction:*

$$-\cfrac{b_0^{(p)}}{a_0^{(p)}(z)-\cfrac{b_1^{(p)}}{a_1^{(p)}(z)-\ddots}},\tag{4.70}$$

where the polynomials $a_j^{(p)}$ *and the numbers* $b_j^{(p)}$ *are given by*

$$b_0^{(p)}=s_1-\alpha s_0,\; b_j^{(p)}=\frac{1}{(d_{j+1}^{\alpha})^2 l_j^{\alpha} l_{j+1}^{\alpha}},\; a_{j-1}^{(p)}=z-\alpha-\frac{1}{l_j^{\alpha}}\left(\frac{1}{d_j^{\alpha}}+\frac{1}{d_{j+1}^{\alpha}}\right),\; j\in\mathbb{N}.\tag{4.71}$$

Proof. Due to (4.68), we obtain $s_0^{(p)}=s_1-\alpha s_0$, and by (2.12), we get $b_0^{(p)}=s_1-\alpha s_0$. By (4.65), the monic Jacobi matrix $J^{(p)}$ is associated with the $J-$fraction (4.70) with atoms $(a_i^{(p)},b_i^{(p)})$ given by (4.71). This completes the proof. □

Corollary 4.6. *Let monic Jacobi matrix* J *admit a* $\mathfrak{L}\mathfrak{U}+\alpha I$ *factorization of the form* (4.60)–(4.62) *and* $J^{(p)}=\mathfrak{U}\mathfrak{L}+\alpha I$ *be its shifted Darboux transformation without parameter. Then the atoms* $(a_i^{(p)},b_i^{(p)})$ *of the* $J-$*fraction associated with* $J^{(p)}$ *can be written in terms of the polynomials of the first kind as*

$$b_j^{(p)}=\frac{b_jP_{j-1}(\alpha)P_{j+1}(\alpha)}{P_j^2(\alpha)}\quad\text{and}\quad a_{j-1}^{(p)}(z)=z-\alpha+\frac{P_j^2(\alpha)+b_jP_{j-1}^2(\alpha)}{P_{j-1}(\alpha)P_j(\alpha)}.\tag{4.72}$$

Proof. By (4.71) and Lemma 3.9, we obtain

$$b_j^{(p)}=\frac{1}{(d_{j+1}^{\alpha})^2 l_j^{\alpha} l_{j+1}^{\alpha}}=\frac{1}{d_{j+1}^{\alpha} l_j^{\alpha} d_{j+1}^{\alpha} l_{j+1}^{\alpha}}=\frac{1}{\dfrac{P_j(\alpha)}{P_{j+1}(\alpha)}\cdot\dfrac{P_j(\alpha)}{b_jP_{j-1}(\alpha)}}=\frac{b_jP_{j-1}(\alpha)P_{j+1}(\alpha)}{P_j^2(\alpha)}.$$

$$\begin{aligned}a_{j-1}^{(p)}&=z-\alpha-\frac{1}{l_j^{\alpha}}\left(\frac{1}{d_j^{\alpha}}+\frac{1}{d_{j+1}^{\alpha}}\right)=z-\alpha-\frac{1}{l_j^{\alpha}}\frac{1}{d_j^{\alpha}}-\frac{1}{l_j^{\alpha}}\frac{1}{d_{j+1}^{\alpha}}\\&=z-\alpha+\frac{P_j(\alpha)}{P_{j-1}(\alpha)}+\frac{b_jP_{j-1}(\alpha)}{P_j(\alpha)}=z-\alpha+\frac{P_j^2(\alpha)+b_jP_{j-1}^2(\alpha)}{P_{j-1}(\alpha)P_j(\alpha)}.\end{aligned}$$

So, (4.72) is proved. This completes the proof. □

5. Shifted Darboux transformation of a monic generalized Jacobs matrices. The case 2×2

Now we study the shifted Darboux transformation of the generalized Jacobi matrices J. We consider the case, where the all the entries of J are 2×2 matrices, i.e., the all normal indices of the sequence $\mathbf{s}$ satisfy the following condition:

$$n_{j+1}-n_j=2.$$

Let us choose $\alpha\in\mathbb{R}$ such that the all polynomials of the first kind P_{n_j} do not vanish at α, i.e.,

$$P_{n_j}(\alpha)\neq 0.\tag{5.73}$$

By (3.29) the generating polynomials $a_j(z) = z^2 + a_1^{(j)} z + a_0^{(j)}$ can be rewritten as

$$
\begin{aligned}
a_0(z) &= z^2 + a_1^{(0)} z - \alpha(a_1^{(0)} + \alpha) - \frac{1}{d_1^\alpha l_1^\alpha} \quad \text{and} \\
a_j(z) &= z^2 + a_1^{(j)} z - \alpha(a_1^{(j)} + \alpha) - \frac{1}{d_{j+1}^\alpha l_j^\alpha} - \frac{1}{d_{j+1}^\alpha l_{j+1}^\alpha},
\end{aligned}
\tag{5.74}
$$

where the numbers d_j^α and l_j^α are calculated by (3.50)– (3.51). Hence, the generalized Jacobi matrix $\mathfrak{J}$ takes the following form:

$$
\mathfrak{J} = \begin{pmatrix} \mathfrak{C}_{a_0} & \mathfrak{D}_1 & \\ \mathfrak{B}_1 & \mathfrak{C}_{a_1} & \ddots \\ & \ddots & \ddots \end{pmatrix}. \tag{5.75}
$$

Due to (3.29) and (5.74), the entries take the form

$$
\begin{aligned}
&\mathfrak{C}_{a_0} = \begin{pmatrix} 0 & 1 \\ \alpha(a_1^{(0)} + \alpha) + \dfrac{1}{d_1^\alpha l_1^\alpha} & -a_1^{(0)} \end{pmatrix}, \ \mathfrak{B}_j = \begin{pmatrix} 0 & 0 \\ \dfrac{1}{d_{j-1} d_j (l_{j-1}^\alpha)^2} & 0 \end{pmatrix}, \\
&\mathfrak{C}_{a_j} = \begin{pmatrix} 0 & 1 \\ \alpha(a_1^{(j)} + \alpha) + \dfrac{1}{d_{j+1}^\alpha l_j^\alpha} + \dfrac{1}{d_{j+1}^\alpha l_{j+1}^\alpha} & -a_1^{(j)} \end{pmatrix} \text{ and } \mathfrak{D}_j = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\end{aligned}
\tag{5.76}
$$

Theorem 5.1. *Let $\mathfrak{J}$ be a monic Jacobi matrix associated with a sequence $\boldsymbol{s} = \{s_i\}_{i=0}^\infty \in \mathcal{H}_{\kappa'}^{k',\alpha-reg}$, and let $\mathcal{N}(\mathbf{s}) = \{n_j\}_{j=0}^\infty$ be a set of normal indices of the sequence $\mathbf{s}$, where $n_{j+1} - n_j = 2$ and $n_0 = 0$. Let P_{n_i} be polynomials of the first kind corresponding to the matrix $\mathfrak{J}$, and let α be some real number such that $P_{n_j}(\alpha) \neq 0$ for all $j \in \mathbb{Z}_+$. Then the monic Jacobi matrix J admits the following $\mathfrak{L}\mathfrak{U}$ – factorization with the shift α:*

$$
\mathfrak{J} = \mathfrak{L}\mathfrak{U} + \alpha I, \tag{5.77}
$$

where I is an infinite identity matrix, and the factorization matrices $\mathfrak{L}$ and $\mathfrak{U}$ are defined by

$$
\mathfrak{L} = \begin{pmatrix} \mathfrak{A}_0 & 0 & \\ \mathfrak{L}_1 & \mathfrak{A}_1 & \ddots \\ & \ddots & \ddots \end{pmatrix} \quad \text{and} \quad \mathfrak{U} = \begin{pmatrix} \mathfrak{U}_0 & \mathfrak{D}_0 & \\ & \mathfrak{U}_1 & \ddots \\ & & \ddots \end{pmatrix}, \tag{5.78}
$$

the entries take the form

$$
\mathfrak{A}_j = \begin{pmatrix} 1 & 0 \\ -a_1^{(j)} - \alpha & 1 \end{pmatrix}, \ \mathfrak{L}_j = \begin{pmatrix} 0 & 0 \\ 0 & \dfrac{1}{d_{j+1}^\alpha l_j} \end{pmatrix}, \ \mathfrak{U}_j = \begin{pmatrix} -\alpha & 1 \\ \dfrac{1}{d_{j+1}^\alpha l_{j+1}} & 0 \end{pmatrix} \text{ and } \mathfrak{D}_j = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{5.79}
$$

where d_j^α and l_j^α are calculated by (3.50)– (3.51).

Proof. Let the sequence $\mathbf{s} \in \mathcal{H}_{\kappa'}^{k',\alpha-reg}$, i.e. there is a real number α such that $P_{n_j}(\alpha) \neq 0$. Then, by [15, Theorem 3.1], Lemma 3.9, and Corollary 3.10, we obtain a $\mathfrak{L}\mathfrak{U}$–factorization of the form (5.77) – (5.80). This completes the proof. □

Now, we construct a shifted Darboux transformation of the generalized Jacobi matrix $\mathfrak{J}$ (5.75)–(5.76). By [15], we get the following statements.

Proposition 5.2. *Let $\boldsymbol{s} = \{s_i\}_{i=0}^\infty \in \mathcal{H}_{\kappa'}^{k',\alpha-reg}$ be associated with a monic Jacobi matrix $\mathfrak{J}$, and let $\mathcal{N}(\mathbf{s}) = \{n_j\}_{j=0}^\infty$ be a set of normal indices of the sequence $\mathbf{s}$, where $n_{j+1} - n_j = 2$ and $n_0 = 0$. Let $\mathfrak{J} = \mathfrak{L}\mathfrak{U} + \alpha I$ be its $\mathfrak{L}\mathfrak{U}$ – factorization with a shift α of the form (4.60)–(4.62). Then we have the following:*

(1) *the shifted Darboux transformation without parameter of* $\mathfrak{J}$ *is the monic Jacobi matrix*

$$\mathfrak{J}^{(p)} = \mathfrak{U}\mathfrak{L} + \alpha I = \begin{pmatrix} -a_1^{(0)} - \alpha & 1 & & & & & \\ \frac{1}{d_1^\alpha l_1^\alpha} & \alpha & 1 & & & & \\ & \frac{1}{d_2^\alpha l_1^\alpha} & -a_1^{(1)} - \alpha & 1 & & & \\ & & \frac{1}{d_2^\alpha l_2^\alpha} & \alpha & 1 & & \\ & & & \frac{1}{d_3^\alpha l_2^\alpha} & -a_1^{(2)} - \alpha & 1 & \\ & & & & \frac{1}{d_3^\alpha l_3^\alpha} & \alpha & \ddots \\ & & & & & \ddots & \ddots \end{pmatrix}. \quad (5.80)$$

(2) *Polynomials of the first and the second kind of the matrix* $\mathfrak{J}^{(p)}$ *can be calculated by*

$$\begin{aligned} &P^{(p)}_{2j-2}(z) = P_{n_{j-1}}(z) \text{ and } P^{(p)}_{2j-1}(z) = \frac{1}{z-\alpha}\left(P_{n_j}(z) + \frac{1}{d_j^\alpha l_j^\alpha} P_{n_{j-1}}(z)\right); \\ &Q^{(p)}_{2j-2}(z) = (z-\alpha)Q_{n_{j-1}}(z) \text{ and } Q^{(p)}_{2j-1}(z) = Q_{n_j}(z) + \frac{1}{d_j^\alpha l_j^\alpha} Q_{n_{j-1}}(z),\ j \in \mathbb{N}. \end{aligned} \quad (5.81)$$

(3) $\mathfrak{J}^{(p)}$ *is associated with a sequence of real numbers* $\mathbf{s}^{(p)} = \{s_i^{(p)}\}_{i=0}^\infty$, *where* $\mathbf{s}^{(p)}$ *is connected with the sequence* $\mathbf{s} = \{s_i\}_{i=0}^\infty$ *as follows:*

$$s_i^{(p)} = s_{i+1} - \alpha s_i, \quad i \in \mathbb{Z}_+ \quad (5.82)$$

Proof. The first statement directly follows from matrix multiplication $\mathfrak{U}$ and $\mathfrak{L}$. We see that $\mathfrak{J}^{(p)} = \mathfrak{U}\mathfrak{L} + \alpha I$ is a shifted Darboux transformation of the matrix $\mathfrak{J}$. By [15, Theorems 3.3-3.4], we get

$$\begin{aligned} &P^{(p)}_{2j-2}(z) = P_{n_{j-1}}(z) \text{ and } Q^{(p)}_{2j-2}(z) = (z-\alpha)Q_{n_{j-1}}(z), \\ &P^{(p)}_{2j-1}(z) = \frac{1}{z-\alpha}\left(P_{n_j}(z) - \frac{P_{n_j}(\alpha)}{P_{n_{j-1}}(\alpha)} P_{n_{j-1}}(z)\right); \\ &\text{and } Q^{(p)}_{2j-1}(z) = Q_{n_j}(z) - \frac{P_{n_j}(\alpha)}{P_{n_{j-1}}(\alpha)} Q_{n_{j-1}}(z),\ j \in \mathbb{N}. \end{aligned} \quad (5.83)$$

Due to Lemma 3.9, $-\frac{P_{n_j}(\alpha)}{P_{n_{j-1}}(\alpha)} = \frac{1}{d_j^\alpha l_j^\alpha}$. Substituting this into (5.83) we obtain (5.81). The third statement follows from [15, Corollary 3.2]. This completes the proof. □

Proposition 5.3. *Let* $\boldsymbol{s} = \{s_i\}_{i=0}^\infty \in \mathcal{H}_{\kappa'}^{k',\alpha-reg}$ *be associated with the monic Jacobi matrix* $\mathfrak{J}$, *and let* $\mathcal{N}(\mathbf{s}) = \{n_j\}_{j=0}^\infty$ *be a set of normal indices of the sequence* $\mathbf{s}$, *where* $n_{j+1} - n_j = 2$ *and* $n_0 = 0$. *Let* $\mathfrak{J} = \mathfrak{L}\mathfrak{U} + \alpha I$ *be its* $\mathfrak{L}\mathfrak{U}$ *– factorization with a shift* α *of the form* (4.60)–(4.62). *Then the shifted Darboux transformation without parameter,* $\mathfrak{J}^{(p)} = \mathfrak{U}\mathfrak{L} + \alpha I$, *is associated with the following* J*–fraction:*

$$-\cfrac{b_0}{a_0^{(p)}(z) - \cfrac{b_1^{(p)}}{a_1^{(p)}(z) - \ddots}}, \quad (5.84)$$

where

$$a_{2i-2}^{(p)}(z) = \frac{m_i^\alpha(z)}{d_i^\alpha}, \quad a_{2i-1}^{(p)}(z) = z - \alpha, \quad b_0^{(p)} = b_0,$$
$$b_{2i-1}^{(p)} = \frac{1}{d_i^\alpha l_i^\alpha} \quad \text{and} \quad b_{2i-2}^{(p)} = \frac{1}{d_{i+1}^\alpha l_i^\alpha}, \quad i \in \mathbb{N}. \tag{5.85}$$

Proof. Due to (5.82), we get

$$s_0^{(p)} = s_1 - \alpha s_0 = \{n_1 = 2 \Rightarrow s_0 = 0,\ s_1 \neq 0 \text{ and } s_1 = b_0\} = b_0. \tag{5.86}$$

Consequently, $b_0^{(p)} = b_0$. By (3.29) and(5.74), we obtain

$$m_i^\alpha(z) = \frac{z + a_1^{(i-1)} + \alpha}{d_i^\alpha}, \quad i \in \mathbb{N}. \tag{5.87}$$

Due to representation (5.80) and (5.86)–(5.87), we obtain that the J-fraction associated with $\mathfrak{J}^{(p)}$ takes the form (5.84). This completes the proof. □

Corollary 5.4. *Let $\boldsymbol{s} = \{s_i\}_{i=0}^\infty \in \mathcal{H}_{\kappa'}^{k',\alpha-reg}$ be associated with a monic Jacobi matrix $\mathfrak{J}$, and let $\mathcal{N}(\mathbf{s}) = \{n_j\}_{j=0}^\infty$ be a set of normal indices of the sequence $\mathbf{s}$, where $n_{j+1} - n_j = 2$ and $n_0 = 0$. Let $\mathfrak{J}$ admit an $\mathfrak{LU}$ – factorization with a shift α of the form (4.60)–(4.62), and let $J^{(p)} = \mathfrak{UL} + \alpha I$ be its shifted Darboux transformation. Then the atoms $(a_i^{(p)}, b_i^{(p)})$ of the $J-$fraction (5.84) associated with $J^{(p)}$ can be rewritten as follows:*

$$a_{2i-2}^{(p)}(z) = \frac{a_{i-1}(z) - a_{i-1}(\alpha)}{z - \alpha}, \quad a_{2i-1}^{(p)}(z) = z - \alpha, \quad b_0^{(p)} = b_0,$$
$$b_{2i-1}^{(p)} = -\frac{P_{n_i}(\alpha)}{P_{n_{i-1}}(\alpha)} \quad \text{and} \quad b_{2i-2}^{(p)} = -\frac{b_i P_{n_{i-1}}(\alpha)}{P_{n_i}(\alpha)}, \quad i \in \mathbb{N}. \tag{5.88}$$

Proof. In view of representation (5.85) and by (3.28), we get

$$a_{2i-2}^{(p)}(z) = \frac{m_i^\alpha(z)}{d_i^\alpha} = \frac{a_{i-1}(z) - a_{i-1}(\alpha)}{z - \alpha}.$$

By Lemma 3.9, $b_i^{(p)}$ in (5.85) can be rewritten as (5.88). This completes this proof. □

Proposition 5.5. *Let $\mathfrak{J}$ be the monic generalized Jacobi matrices defined by (5.75)–(5.76) and let $\mathfrak{J}^{(p)} = \mathfrak{UL} + \alpha I$ be the shifted Darboux transformation of J. Then the $\mathbf{m}$–function of $\mathfrak{J}^{(p)}$ is given by*

$$m_{[0,i-1]}^{(p)}(z) = \frac{(z-\alpha)Q_i^+(z,\alpha)}{P_i^+(z,\alpha)}. \tag{5.89}$$

Furthermore, $m_{[0,i-1]}^{(p)}$ admits the following asymptotic expansion:

$$m_{[0,i-1]}^{(p)}(z) = -\frac{s_1}{z} - \frac{s_2 - \alpha s_1}{z^2} - \cdots - \frac{s_{2i-1} - \alpha s_{2i-2}}{z^{2i-1}} + o\left(\frac{1}{z^{2i-1}}\right), \quad z\widehat{\to}\infty. \tag{5.90}$$

Proof. Let us prove (5.89). In view of the representation matrix, we consider two cases. Suppose $i := 2j - 1$ and $j \in \mathbb{N}$. Then

$$m_{[0,i-1]}^{(p)}(z) = m_{[0,2j-2]}^{(p)}(z) = -\frac{Q_{2j-1}^{(p)}(z)}{P_{2j-1}^{(p)}(z)} = \{\text{by (3.26) and (5.83)}\} = \frac{(z-\alpha)Q_{2j-1}^+(z,\alpha)}{P_{2j-1}^+(z,\alpha)}.$$

Assume $i := 2j$ and $j \in \mathbb{N}$. Then

$$m_{[0,i-1]}^{(p)}(z) = m_{[0,2j-1]}^{(p)}(z) = -\frac{Q_{2j-1}^{(p)}(z)}{P_{2j-1}^{(p)}(z)} = \{\text{by (3.26) and (5.81)}\} = \frac{(z-\alpha)Q_{2j}^+(z,\alpha)}{P_{2j}^+(z,\alpha)}.$$

So, (5.89) is proved. By (2.22) and (5.82), we obtain that (5.90) holds. □

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Received 02/02/2021; Revised 21/02/2021