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On some methods of studying the structure of group rings of groups of finite rank

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Анотація. У роботі розглянуті деякі нові методи вивчення структури групових кілець абелевих груп скінченого рангу. Ці методи базуються на подальшому розвитку деяких результатів теорії полів, таких як теорія Куммера і теорема Діріхле.

Abstract. In the paper we consider some new methods available for studying of the structure of group rings of abelian groups of finite rank. These methods are based on further development of some results of the theory of fields such as Kummer theory and Dirichlet theorem.

In the paper we use the standard denotations of the theory of fields (see [**8**]).

Let A be an abelian group and let B be a subgroup of the group A. The set $is_A(B)$ of elements $a \in A$ such that $a^n \in B$ for some positive integer n is a subgroup of the group A which s named as the isolator of the subgroup B in the group A. The subgroup B is said to be dense in the group A if $is_A(B) = A$. If $is_A(B) = B$ then the subgroup B is said to be isolated in the group A. As usually, $t(A)$ denotes the torsion subgroup of the group A and $\pi(A)$ denotes the set of prime divisors of orders of elements of the group A if the group A is torsion.

Let A be a torsion-free abelian group of finite rank acted by a groups Γ. Elements of the group A which have finite orbits under action of the group Γ form a subgroup $\Delta_\Gamma(A)$ of the group A. Let p be a prime number, we denote by $\Lambda^p_\Gamma(A)$ the isolator in the group A of a subgroup generated by all elements $a \in A$ for which the group Γ has a subgroup Γ_a of finite index such that any element $\gamma \in \Gamma_a$ acts on the element a in the way $a^\gamma = a^{p^m}$ for some integer number m.

The group A is said to be a Γ-plinth if $A \bigotimes_{\mathbb{Z}} \mathbb{Q}$ is an irreducible $\mathbb{Q}\Gamma_1$-module for any subgroup $\Gamma_1 \leq \Gamma$ of finite index.

Let R be a ring, let G be a group and let I be a right ideal of the group ring RG. The ideal I is said to be faithful if $I^\dagger = \quad G \cap (1 + I) = 1$. We say that a subgroup H of the group G controls the ideal I if

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$$I = (I \cap kH)kG. \tag{1}$$

The intersection $c(I)$ of all subgroups of the group G controlling the ideal I is said to be the controller of the ideal I.

Let H be a subgroup of the group G and let U be a right RH-module. Since the group ring RG can be considered as a left RH-module, we can define the tensor product $U \otimes_{RH} RG$ which is a right RG-module named as the RG-module induced from the RH-module U. Besides,

$$M = U \otimes_{RH} RG \tag{2}$$

if and only if

$$M = \oplus_{t \in T} Ut, \tag{3}$$

where T is a right transversal to the subgroup H in the group G.

Suppose that $M = aRG$ is a cyclic RG-module generated by a nonzero element $a \in M$. Put $I = Ann_{kG}(a)$ and let $U = akH$, where H is a subgroup of the group G. It is not difficult to note that, in these denotations, the identity (1) holds if and only if the identity (3) holds. Thus, in this case, the identities (1), (2) and (3) mean the same.

Intensive investigations of group algebras of torsion-free abelian groups of finite rank were stimulated by a famous problem of Ph. Hall: whether any irreducible representation of a polycyclic group over a locally finite field must be finite dimensional. There were many unsuccessful attempts of solving this problem until J. Roseblade answered it affirmatively . Fortunately, at that time a very powerful tool for solving of the problem have already existed. This tool became well known as a conjecture of A.E. Zalessky proved by G.Bergman in [**1**].

The conjecture of A. E. Zalessky states that if A is a polycyclic (i.e. finitely generated as an abelian group) Γ-plinth and P is a maximal Γ-invariant ideal of the group algebra $\Bbbk A$, where $\Bbbk$ is a field, then $dim_{\Bbbk}\Bbbk A/P$ is finite (we should note that originally the conjecture was formulated in some different but equivalent way [**6**]).

The Bergman's proof was based on the theory of valuations of fields. The main prerequisite for this approach is the following: if P is a prime ideal of the group algebra $\Bbbk A$ then the quotient ring $\Bbbk A/P$ is a domain. Certainly, properties of the domain $\Bbbk A/P$ are strongly related with the properties of the ideal P and studying $\Bbbk A/P$ we can obtain some results on P. As $\Bbbk A/P$ is a domain, it is included in some field $\mathbb{K}$ and it becomes possible to use some methods of the theory of fields. Bergman applied valuations defined on $\mathbb{K}$ for studying $\Bbbk A/P$. This approach is also describe by D. Passman in [**9**].

The results and methods of Bergman were deeply developed by Riseblade in [**10**] where he proved that if A is a finitely generated torsion-free abelian group acted by a groups Γ

and P is a faithful prime ideal of the group algebra $\Bbbk A$ such that $|\Gamma : N_\Gamma(P)| < \infty$ then P is controlled by $\Delta_\Gamma(A)$ (see theorem D of [**10**]).

Let $\Bbbk$ be a field, A be a torsion-free abelian group of finite rank acted by a group of operators Γ and $\Bbbk$ be a field. Let I be an ideal of the group algebra $\Bbbk A$. A subgroup $S_\Gamma(I)$ of the group Γ which consists of elements $\gamma \in \Gamma$ such that $I \cap \Bbbk B = I^\gamma \cap \Bbbk B$ for some finitely generated dense subgroup B of the group A is said to be the standardizer of the ideal I in the group Γ (see [**5**]).

Theorem A of a Brookes paper [**5**] states that if P is a faithful prime ideal of the group algebra $\Bbbk A$ such that $S_\Gamma(P) = \Gamma$ then the ideal P is controlled by the subgroup $\Delta_\Gamma(A)$. In the case, where the group A is finitely generated, the result follows from theorem D of [**10**].

We should note that theorem A of [**5**] was referenced in many papers and was considered as the key result for studying of group rings of soluble groups of finite rank and their applications. There even appeared some generalizations of this theorem. However, as it became known in the beginning of the century, the original proof of theorem A of [**5**] is incorrect. Moreover, there is an example which shows that in fact the theorem does not hold in the case, where the field $\Bbbk$ has positive characteristic (see [**17**], §3, Example). The original proof of theorem A of [**5**] is based on the method developed by Roseblade in [**10**] and also is dealing with the theory of valuations only. The error occurred in the proof of lemma 10.2 from [**5**] where equivalent valuations of the field $\Bbbk(A)$ were interpreted as equivalent functions of the multiplicative group $\Bbbk(A)^*$ of the field $\Bbbk(A)$. However, the equivalence of valuations means only that they have the same ring of valuations and does not mean that these valuations coincide as functions (see [**18**], Chap. VI, §8). This inaccuracy lead to a mistaken conclusion that a finite set of valuations from the formula (32) of [**5**] induces a finite set of homomorphism θ_v of the group A, although in reality this set is infinite. It entailed a fallacious formula "$\vartheta_v(a) = \vartheta_v(a^\gamma)$" (see [**5**], p. 47, the row 5) which brought the erroneous assertion of lemma 10.2 from [**5**].

In fact the example of [**17**], §3 shows that methods based on valuations only does not allow us to spread the results of [**10**] to the case of group algebras of torsion-free abelian groups of finite rank. So, we need to develop new methods and approaches.

The main idea of our studying is that in the case, where P is a faithful prime ideal of the group algebra $\Bbbk A$, the quotient ring $\Bbbk A/P$ can be embedded as a domain $\Bbbk[A]$ in a field F and, as the ideal P is faithful, the group A becomes a subgroup of the multiplicative group of the field F. It allows us to apply for studying of $\Bbbk[A]$ some methods of the theory of fields such as Kummer theory and properties of the multiplicative groups of fields. For the first time, such methods were applied in [**12**] (see [**12**], lemmas 2, 5) where we proved the identity (3) for modules over abelian groups of finite rank. Then the methods were developed in [**11**],[**13**], [**14**], [**15**],[**16**] and [**17**]. In its tern properties of the quotient ring $\Bbbk[A] \cong \Bbbk A/P$ strongly depend on the properties of the ideal P. To prove identity (1) we prove identities (2) and (3) for a $\Bbbk A$-module $\Bbbk[A]$.

Let $\Bbbk$ be a subfield of a field f and let G be a subgroup of the multiplicative group f^* of the field f. Then the field $\Bbbk(G)$ may be considered as a $\Bbbk G$-module and the field $\Bbbk$ can be considered as a $\Bbbk(G \cap \Bbbk^*)$-module. Therefore, we can define the tensor product $\Bbbk \otimes_{\Bbbk(G\cap\Bbbk^*)} \Bbbk G$ and the equation $\Bbbk(G) = \Bbbk \otimes_{\Bbbk(G\cap\Bbbk^*)} \Bbbk G$ means that $\Bbbk(G) = \oplus_{t\in T}\Bbbk t$, where T is a transversal to the subgroup $G \cap \Bbbk*$ in the group G. If $|G/G \cap \Bbbk^*| = m < \infty$ the equation $\Bbbk(G) = \Bbbk \otimes_{\Bbbk(G\cap\Bbbk^*)} \Bbbk G$ holds if and only if $[\Bbbk(G) : \Bbbk] = m$. The relations between $|G/G \cap \Bbbk^*|$ and $[\Bbbk(G) : \Bbbk]$ were considered in Kummer theory (see [**8**], Chap. VIII, theorem 13).

Theorem 1 ([**16**], theorem 2.3)**.** *Let $\Bbbk$ be a subfield of a field f and let G be a subgroup of the multiplicative group f^* of the field f such that the quotient group $G\Bbbk^*/\Bbbk^*$ is torsion and such that char$\Bbbk \notin \pi(G\Bbbk^*/\Bbbk^*)$, for any prime number $p \in (\pi(t(G\Bbbk^*)) \cap \pi(G\Bbbk^*/\Bbbk^*))$ the subfield $\Bbbk$ contains a primitive root of degree p from 1 and the subfield $\Bbbk$ contains a primitive root of degree 4 from 1 if the quotient group $G\Bbbk^*/\Bbbk^*$ contains an element of order 4. Then $\Bbbk(G) = \Bbbk \otimes_{\Bbbk(\Bbbk^*\cap G)} \Bbbk G{=}\oplus_{t\in T}\Bbbk t$, where T is a transversal of the subgroup $\Bbbk^* \cap G$ in the group G*

Corollary 1 ([**16**], corollary 2.4)**.** *Let $\Bbbk$ be a subfield of a field f and suppose that the subfield $\Bbbk$ contains all roots from 1. Let G be a subgroup of the multiplicative group f^* of the field f and let $B = \Bbbk^* \cap G$. Suppose that the quotient group G/B is torsion and char$\Bbbk \notin \pi(G/B)$. Then $\Bbbk(A) = \Bbbk \otimes_{\Bbbk B} \Bbbk A = \oplus_{t\in T}\Bbbk t$, where T is a transversal of the subgroup B in the group G*

Theorem 1 can be considered as a generalization of theorem 13 of [**8**], Chap. VIII to the case of infinitely dimensional extensions.

An abelian group is said to be minimax if it has a finite series each of whose factor is either cyclic or quasi-cyclic. If A is an abelian minimax group then the spectrum $Sp(A)$ of the group A is the set of prime numbers p such that the group A has an infinite p-section. It is easy to note that the set $Sp(A)$ is finite.

A field $\Bbbk$ is said to be regular if it is countable and the multiplicative group of the field $\Bbbk$ is the direct product of a torsion and a free abelian groups. A field $\Bbbk$ is said to be finitely generated if it is obtained by joining of a finite set to the minimal subfield of $\Bbbk$. The following result on the construction of the multiplicative group of some fields is an important tool for our further investigations.

Lemma 1 ([**16**], proposition 3.3 and [**17**], proposition 1.1.4)**.** *Let f be an algebraically closed field and let $\Bbbk$ be a finitely generated subfield of the field f. Let π be a finite set* *of prime numbers such that* $\operatorname{char} f \notin \pi$, *let* X *be the set of all roots from 1 in the field* f *and let* X_π *be the set of all roots from 1 of degree* q^n, *where* $n \in \mathbb{N}$ *and* $q \in \pi$. *Let* p *be a prime number and let* A *be a minimax torsion-free subgroup of the multiplicative group* f^* *of the field* f *such that* $Sp(A) \subseteq \{p\}$ *and let* B *be a finitely generated dense subgroup of the group* A *such that the quotient group* A/B*is a* p*-group. Then:*

(i) the field $\Bbbk$ *is regular and the group* $t(\Bbbk^*)$ *is finite;*

(ii) the field $\Bbbk(X_\pi)$ *is regular and the group* $t((\Bbbk(X_\pi))^*)$ *is locally cyclic Chernikov ;*

(iii) if $\operatorname{char} f = p, s = \Bbbk(X)(B)$ *and* $h = \Bbbk(X)(A)$ *then the field* s *is regular and the quotient group* h^*/s^* *is a* p *-group.*

Applying of theorem 1 and lemma 1 allows to prove the following theorem.

Theorem 2. *Let* $\Bbbk$ *be a finitely generated field, let* A *be an abelian torsion-free group of finite rank and let* P *be a faithful prime ideal of the group algebra* $\Bbbk A$. *Then:*

*(i) ([**16**], theorem 4.1) if the field* $\Bbbk$ *has characteristic zero then the controller* $c(P)$ *of the ideal* P *is a finitely generated subgroup of the group* A *;*

*(ii) ([**17**],theorem 2.1.2(iv))if the field* $\Bbbk$ *has nonzero characteristic* p *then the controller* $c(P)$ *of the ideal* P *is a minimax subgroup of the group* A *such that* $Sp(c(P)) \subseteq \{p\}$

In theorem 3 we proved that if $\operatorname{char}\Bbbk = p$ then in theorem A of [**5**] the subgroup $\Delta_\Gamma(A)$ can be replaced by the subgroup $\Lambda_\Gamma^p(A)$ and theorem 2 plays the key role in the proof.

Theorem 3. *Let* $\Bbbk$ *be a field, let* A *be a torsion-free abelian group of finite rank acted by a group of operators* Γ *and let* P *be a faithful prime ideal of the group algebra* $\Bbbk A$. *Suppose that* $S_\Gamma(P) = \Gamma$. *Then:*

*(i) ([**16**], theorem 5.4) if the field* $\Bbbk$ *has characteristic zero then the ideal* P *is controlled by the subgroup* $\Delta_\Gamma(A)$*;*

*(ii) ([**17**], theorem 3.3.7) if the field* $\Bbbk$ *has nonzero characteristic* p *then the ideal* P *is controlled by the subgroup* $\Lambda_\Gamma^p(A)$.

At the present time any other approach to the proof of the new version of the Brookes theorem A from [**5**] has not been published, yet. So, in the present situation, any independent proof of the Brookes theorem could be quite topical and helpful for reproving of various results where theorem A from [**5**] is referenced.

We also should note that there are very interesting original methods developed by R. Bieri and R. Strebel in [**2**] and by R. Bieri and R. J. Groves in [**3**], [**4**] which allow to obtain deep results on the structure of group algebras of torsion-free finitely generated abelian groups. So, it would be very interesting to spread these methods on the case of group algebras of torsion-free abelian groups of finite rank.

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