Науковий часопис НПУ імені М.П.Драгоманова. Серія 1. Фізико-математичні науки
Київ: НПУ імені М.П. Драгоманова. — 2013, № 14.— С. 14–23.

Differential graded categories associated with the Dynkin diagrams

O. Gnatiuk, N. Golovaschuk

Taras Shevchenko National University of Kyiv

Анотація. В даній роботі розглядаються диференціальні градуйовані категорії, асоційовані з градуйованими графами, які мають додатну квадратичну форму. Для таких задач розв'язується класифікаційна задача, а саме, наводиться алгоритм перетворень і показується, що задачі з розглядуваного класу можуть бути перетворені до задач, граф яких є колчаном типу Динкіна.

Abstract. This work concerns with differential graded categories associated with graded graphs with positive quadratic form. We solve the classification problem for such differential graded categories. Those problems can be transformed to the problems with graded graph, which is a quiver of Dynkin type. The algorithm is built.

Preliminaries

The reduction algorithm of linear categories and other structures is widely used in the representation theory. This approach allows to study representations inductively, reducing the corresponding categories step by step ([**1**]). On the other hand, the important characteristic of represented structure is the induced quadratic form whose roots correspond to the indecomposable representations. The theory of quadratic forms is well known ([**2**], [**3**], [**4**]). We give the simultaneous reduction algorithm of transformation of the differential graded category with special properties and the underlined unit quadratic form to the canonical form.

1. Differential graded categories and directed graded graph

The $\Bbbk$-linear category $\mathcal{U}$ is called *graded* if $\mathcal{U}(\mathtt{i},\mathtt{j}) = \oplus_{q\in\mathbb{Z}}\mathcal{U}_q(\mathtt{i},\mathtt{j})$ is a sum of finite dimensional vector spaces $\mathcal{U}_q(\mathtt{i},\mathtt{j}) = \deg^{-1}(q)$, $\mathtt{i},\mathtt{j} \in \mathrm{Ob}\,\mathcal{U}$. The graded $\Bbbk$-category $\mathcal{U}$ is called the *differential graded category* or *dgc* if there is the differential $\mathtt{d}: \mathcal{U} \to \mathcal{U}$ which maps $\mathtt{d}: \mathcal{U}_q(\mathtt{i},\mathtt{j}) \to \mathcal{U}_{q+1}(\mathtt{i},\mathtt{j})$, $q \in \mathbb{Z}$, $\mathtt{i},\mathtt{j} \in \mathrm{Ob}\,\mathcal{U}$, and the following properties hold:

(1) $\mathtt{d}(1_{\mathtt{i}}) = 0$, $\mathtt{i} \in \mathrm{Ob}\,\mathcal{U}$;

E-mail: olena.gnatyuk@gmail.com

(2) Leibnitz rule: $\mathtt{d}(x_1 \dots x_{i-1} x_i \dots x_k) =$
$= \sum_{i=1}^{k} \hat{x}_1 \dots \hat{x}_{i-1} \mathtt{d}(x_i) x_{i+1} \dots x_k = \sum_{i=1}^{k} (-1)^{|x_1|} x_1 \dots (-1)^{|x_i|} x_i x_{i+1} \dots x_k$;

(3) $\mathtt{d}^2 = 0$.

Let $\Gamma = (\Gamma_0, \Gamma_1, \mathtt{s}, \mathtt{t})$ be a *directed* graph with Γ_0 be a set of vertices and Γ_1 be a set of edges (arrows) equipped with two maps $\mathtt{s} : \Gamma_1 \to \Gamma_0$ and $\mathtt{t} : \Gamma_1 \to \Gamma_0$ that return starting and end (terminating) vertex of the edge correspondingly. Two vertices $\mathtt{i}, \mathtt{j} \in \Gamma_0$ are called *incident* if $\Gamma_1(\mathtt{i}, \mathtt{j}) \cup \Gamma_1(\mathtt{j}, \mathtt{i}) \neq \varnothing$. The graph $\Gamma = (\Gamma_0, \Gamma_1, \mathtt{s}, \mathtt{t})$ is called *graded* (or $\mathbb{Z}$-graded) if there is the map $\deg : \Gamma_1 \to \mathbb{Z}$, such that

$$\Gamma_1^q = \bigsqcup_{\mathtt{i}, \mathtt{j} \in \Gamma_0} \Gamma_1^q(\mathtt{i}, \mathtt{j}) = \deg^{-1}(q), \quad \Gamma_1 = \bigsqcup_{q \in \mathbb{Z}} \Gamma_1^q.$$

We denote $|x| = \deg x$ and $\hat{x} = (-1)^{|x|} x$. The graph Γ is called 0-quiver or *quiver* if $\Gamma_1^q(\mathtt{i}, \mathtt{j}) = \varnothing$ whenever $q \neq 0$.

Let $\Bbbk$ be an algebraically closed field. We consider $\Bbbk\Gamma$ the $\Bbbk$-linear path category of the graded graph Γ which is freely generated over $\Bbbk$ by all the pathes on Γ. We denote $\operatorname{coeff}_{x_1 \dots x_k} x = \kappa$, $\kappa \in \Bbbk$ whenever $x = \kappa x_1 \dots x_k + \dots$ is a basis decomposition. The category $\Bbbk\Gamma$ inherits the degree (graduation) from Γ such that $\deg x_1 x_2 \dots x_k = \sum_{i=1}^{k} \deg x_i$.

The full subgraph Γ_S, $S \subset \Gamma_0$ is called *closed contour* if there is an ordering $S = \{\mathtt{i}_1, \dots, \mathtt{i}_k\}$ such that $|\Gamma_1(\mathtt{i}_j, \mathtt{i}_{j+1}) \cup \Gamma_1(\mathtt{i}_{j+1}, \mathtt{i}_j)| > 0$, $\quad j = 1, \dots, k-1$, and $|\Gamma_1(\mathtt{i}_1, \mathtt{i}_k) \cup \Gamma_1(\mathtt{i}_k, \mathtt{i}_1)| > 0$. The closed contour Γ_S, $S = \{\mathtt{i}_1, \dots, \mathtt{i}_k\} \subset \Gamma_0$ is called *clear* if $\Gamma_1(\mathtt{i}_s, \mathtt{i}_t) \cup \Gamma_1(\mathtt{i}_t, \mathtt{i}_s) = \varnothing$, $\quad |s - t| > 1 \pmod{k}$. The closed contour Γ_S is called *oriented cycle* if $|\Gamma_1(\mathtt{i}_j, \mathtt{i}_{j+1})| > 0$, $\quad j = 1, \dots, k-1$, and $|\Gamma_1(\mathtt{i}_k, \mathtt{i}_1)| > 0$. The closed contour Γ_S is called *detour contour* if $|\Gamma_1(\mathtt{i}_j, \mathtt{i}_{j+1})| > 0$, $\quad j = 1, \dots, k-1$, and $|\Gamma_1(\mathtt{i}_1, \mathtt{i}_k)| > 0$. Denote $x_{\mathtt{ij}}$ the edge from the vertice starting in $\mathtt{i}$ and ending in $\mathtt{j}$. Detour contour Γ_S is called *active* (or contour of differenrial type) if $\kappa x_{\mathtt{i}_1 \mathtt{i}_2} \dots x_{\mathtt{i}_{k-1} \mathtt{i}_k}$ is a summand of differential of the edge $x_{\mathtt{i}_1 \mathtt{i}_k}$. The edge $a \in \Gamma_1(\mathtt{i}, \mathtt{j})$ is called *deep* if there are no other pathes on Γ from $\mathtt{i}$ to $\mathtt{j}$. The edge $a \in \Gamma_1(\mathtt{i}, \mathtt{j})$ is called *minimal* if $\mathtt{d}(a) = 0$.

Given a dgc $\mathcal{U}$ with $|\mathrm{Ob}\,\mathcal{U}| < \infty$, define the underlined directed graded graph $\Gamma = \Gamma(\mathcal{U})$ such that $\Gamma_0 = \mathrm{Ob}\,\mathcal{U}$, and $\Gamma_1(\mathtt{i}, \mathtt{j})$ is a basis of $(\mathcal{U}/\mathcal{U}^{\otimes 2})(\mathtt{i}, \mathtt{j})$, $\mathtt{i}, \mathtt{j} \in \Gamma_0$ with the induced graduation. The differential $\mathtt{d}$ induces the map $\mathtt{d} : \Gamma_1^q \to \Bbbk\Gamma_{q+1}(\mathtt{i}, \mathtt{j})$, $\quad \mathtt{i}, \mathtt{j} \in \Gamma_0$, $\; q \in \mathbb{Z}$. which is extended on the whole $\Bbbk\Gamma$ by Leibnitz rule.

The graph Γ which is correspondent to the finite dimensional differential graded category is finite. The graph Γ is called *correctly defined* if it has no oriented cycles and it does not have multiple edges. In this case $\Bbbk\Gamma$ is finitely generated.

2. Quadratic form

We associate with correctly defined graded graph $\Gamma = (\Gamma_0, \Gamma_1, \mathtt{s}, \mathtt{t})$ the undirected bigraph $\mathcal{B} = \mathcal{B}(\Gamma) = (\Gamma_0, \mathcal{B}_1)$ in the following way. We denote by $\mathcal{B}_1$ the set of pairs $\{\mathtt{i}, \mathtt{j}\}$ of vertices from Γ_0 that are incident in Γ together with correspondent to Γ graduation $\deg(\{\mathtt{i}, \mathtt{j}\}) = |\{\mathtt{i}, \mathtt{j}\}| = \deg a \pmod 2$, $a \in \Gamma_1(\mathtt{i}, \mathtt{j})$, then $\mathcal{B}_1 = \mathcal{B}_1^0 \sqcup \mathcal{B}_1^1$. Here $\mathcal{B}_1^0$ is a set of undirected edges of degree 0 and $\mathcal{B}_1^1$ is a set of undirected edges of degree 1. Denote by $\chi = \chi(\Gamma)$ the integral unit quadratic form such that $\chi : \mathbb{Z}^n \to \mathbb{Z}$,

$$\chi(x) = \sum\nolimits_{\mathtt{i} \in \Gamma_0} x_{\mathtt{i}}^2 - \sum\nolimits_{\{\mathtt{i}, \mathtt{j}\} \in \mathcal{B}_1} (-1)^{|\{\mathtt{i}, \mathtt{j}\}|} x_{\mathtt{i}} x_{\mathtt{j}}.$$

For the graph $\Gamma = (\Gamma_0, \Gamma_1, \mathtt{s}, \mathtt{t})$ and $i, j \in \Gamma_0$ we denote by (i, j) — the edge of graph Γ with unknown or arbitrary direction. The edges with even degree are usually drown solid and the edges with odd degree are drown dotted.

We say that χ is *positive* if $\chi(r) > 0$ for all $r \neq 0$. An integer vector $r \in \mathbb{Z}^n$ is called a *root* if $\chi(r) = 1$. The canonical base vectors $\mathtt{e}^{\mathtt{i}}$ are called simple roots. The root $r = \sum_{\mathtt{i} \in \Gamma_0} r_i \mathtt{e}^{\mathtt{i}}$ is called *positive root* (resp., negative root) if in addition $r_{\mathtt{i}} \in \mathbb{Z}_+$ (resp., $r_{\mathtt{i}} \in \mathbb{Z}_-$) for any $\mathtt{i} \in \Gamma_0$ (we assume $0 \in \mathbb{Z}_+ \cap \mathbb{Z}_-$). The root r is called *sincere* if $r_{\mathtt{i}} \neq 0$ for all $\mathtt{i} \in \Gamma_0$. Two integral forms $\chi, \chi' : \mathbb{Z}^n \to \mathbb{Z}$ are $\mathbb{Z}$-*equivalent* if they describe the same maps up to a change of basis, that is, if there exists a linear $\mathbb{Z}$-invertible transformation $T : \mathbb{Z}^n \to \mathbb{Z}^n$ such that $\chi' = \chi T$.

For $\{\mathtt{i}, \mathtt{j}\} \in \mathcal{B}_1$, we denote by $T_{\mathtt{ij}}^{\varepsilon} : \mathbb{Z}^n \to \mathbb{Z}^n$ the $\mathbb{Z}$-linear transformation ([**4**], [**5**]):

$$T_{\mathtt{ij}}^{\varepsilon}(\mathtt{e}^t) = \begin{cases} \mathtt{e}^t, & \text{if } t \neq i; \\ \mathtt{e}^i + (-1)^{|\{\mathtt{i}, \mathtt{j}\}|} \mathtt{e}^j, & \text{if } t = i. \end{cases} \tag{1}$$

with $\varepsilon = (-1)^{|\{\mathtt{i}, \mathtt{j}\}|} \in \{+, -\}$. If a degree $|\{\mathtt{i}, \mathtt{j}\}|$ is even then we call $T_{\mathtt{ij}}^{+}$ an *inflation* for χ, if $|\{\mathtt{i}, \mathtt{j}\}|$ is odd, we call $T_{\mathtt{ij}}^{-}$ a *deflation* for χ. The forms χ and $\chi T_{\mathtt{ij}}^{\varepsilon}$ are $\mathbb{Z}$-equivalent, if χ is a unit form, then $\chi' = \chi T_{\mathtt{ij}}^{\varepsilon}$ is a unit form, and χ is positive if and only if χ' is positive.

For bigraph $\mathcal{B}$ we will use notions of *chain*, *simple and closed chain*, *tree* and *forest* in common meaning. We say that tree $\mathcal{B}$ is 0-tree (0-forrest) if any edge has degree 0. Any point of tree which is incident with more than two edges is called *branch point.* If point x is not branch and $\mathcal{B}|_S$ connected component of $\mathcal{B}|_{\Gamma_0 \setminus \{x\}}$ which does not contain branch point then the full subgraph $\mathcal{B}|_{S \cup \{x\}}$ is called *tail* of x and is denoted by $\overrightarrow{x}$.

Proposition 1 ([**4**])**.** *Let χ be an integral positive unit form, $\mathcal{B}$ its bigraph. Then there is a sequence of deflations of type* (1) *with composition T such that the bigraph $\mathcal{B}T$ of form χT is a 0-forrest of Dynkin type. In this case, it is a disjoint union of some of the following Dynkin diagrams: A_n $(n \geqslant 1)$, D_n $(n \geqslant 4)$, or E_n $(n = 6, 7, 8)$. If $\mathcal{B}$ is connected then $\mathcal{B}T$ is just a 0-tree. The Dynkin type is uniquely defined by χ.*

We say that a graph Γ is *reduced* (*A-reduced*) if its underlined bigraph $\mathcal{B}(\Gamma)$ can be reduced to a disjoint union of Dynkin diagrams (Dynkin diagram of A type).

3. The main result

We consider the problems, that consist of the differential graded category (dgc) $\mathcal{U}$ together with it's underlined directed graded graph Γ and undirected bigraph $\mathcal{B}$. We consider only dgc each clear contour of which is active and underlined graph of which is correctly defined. Such problem is denoted by $(\mathcal{U}, \Gamma, \mathcal{B})$. The class of such problems is denoted by Υ.

The connected problem $(\mathcal{U}, \Gamma, \mathcal{B}) \in \Upsilon$ is called Dynkin problem and the correspondent graph Γ is called Dynkin directed graded graph if $\mathcal{B}(\Gamma)$ is one of the Dynkin diagrams $(A_n, D_n, E_6, E_7, E_8)$. If $\mathcal{B}(\Gamma) = A_n$ then we say that Γ is A_n-graph and analogically for all types A_n, D_n, E_6, E_7, E_8.

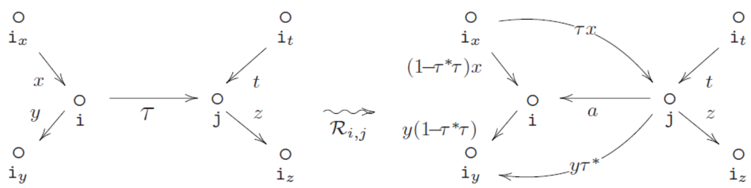
Theorem 1. *Let $\mathcal{U}$ be differential graded category having a correctly defined underlined graded graph Γ, and the quadratic form χ is positively defined. We assume that any clear contour is a contour of differential type. Then there exists a composition of reductions $\mathcal{R} : \mathcal{U} \to \mathcal{U}'$ such that $\Gamma_{\mathcal{U}'}$ is a disjoint union of graphs of Dynkin type.*

4. Reduction and preliminary Lemma

We consider a problem $(\mathcal{U}, \Gamma, Q) \in \Upsilon$. The algorithm of reduction of the problem $(\mathcal{U}, \Gamma, \mathcal{B})$ is shown in [**5**]. We will describe those action on graph Γ, this represents those algorithm on the whole problem $(\mathcal{U}, \Gamma, \mathcal{B})$.

Here on the diagrams below we draw all edges as solid arrows but they can have different degrees, moreover, we depict the direction of the arrow, if it does not matter.

Suppose that $\tau \in \Gamma_1(\mathtt{i}, \mathtt{j})$ is a deep minimal regular edge with degree $\deg \tau = |\tau|$. The general case is:



Define the reduction on $\mathcal{R}_{\mathtt{ij}}(\Gamma)$. We assume that there is $\tau^* : \mathtt{j} \to \mathtt{i}$ such that $\tau\tau^* = 1_{\mathtt{j}}$, and $1_{\mathtt{i}} = 1_{\mathtt{i}_1} + 1_{\mathtt{i}_2} = (1 - \tau^*\tau) + \tau^*\tau$ is a decomposition on the sum of mutually commuting idempotents. For any $x : \mathtt{i}_x \to \mathtt{i}$ we obtain the edges $(1-\tau^*\tau)x : \mathtt{i}_x \to \mathtt{i}$, $|(1-\tau^*\tau)x| = |x|$ and $\tau x : \mathtt{i}_x \to \mathtt{j}$, $|\tau x| = |x| + |\tau|$, besides, $\mathtt{d}'((1-\tau^*\tau)x) = a\tau x + (\mathtt{d}(x))'$. For any $y : \mathtt{i} \to \mathtt{i}_y$ there are: $y(1-\tau^*\tau) : \mathtt{i} \to \mathtt{i}_y$, $|y(1-\tau^*\tau)| = |y|$ and $y\tau^* : \mathtt{i} \to \mathtt{j}$, $|y\tau^*| = |y| - |\tau|$, and, $\mathtt{d}'(y\tau^*) = y(1-\tau^*\tau)a + (\mathtt{d}(y))'$.

The differential on $\mathcal{R}_{\mathtt{ij}}\mathcal{U}$ is obtained by substitution $\mathtt{1_i} = (1-\tau^*\tau) + \tau^*\tau$. Then any path crossing on the point $\mathtt{i}$ is a combination of two pathes:

$$y_1 \dots y_q \, yx \, x_p \dots x_1 \iff y_1 \dots y_q \; \big(y(1-\tau^*\tau) \;\; y\tau^*\tau\big) \begin{pmatrix} (1-\tau^*\tau)x \\ \tau^*\tau x \end{pmatrix} x_p \dots x_1.$$

Lemma 1. *Let $(\mathcal{U}, \Gamma, \mathcal{B}) \in \Upsilon$. Let $\tau \in \Gamma_1(\mathtt{i}, \mathtt{j})$ be a minimal deep regular edge, and let $\mathcal{R}_{\mathtt{ij}} : \mathcal{U} \to \mathcal{U}'$ be a complete reduction. Then $(\mathcal{R}_{\mathtt{ij}}\mathcal{U}, \mathcal{R}_{\mathtt{ij}}\Gamma, \mathcal{R}_{\mathtt{ij}}\mathcal{B}) \in \Upsilon$.*

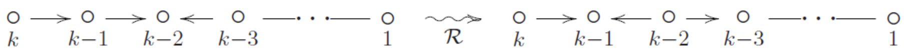
We denote the reduced problem $(\mathcal{R}_{\mathtt{ij}}\mathcal{U}, \mathcal{R}_{\mathtt{ij}}\Gamma, \mathcal{R}_{\mathtt{ij}}\mathcal{B})$ simply by $\mathcal{R}_{\mathtt{ij}}\Gamma$. The composition of reductions $\mathcal{R}_{i_1,j_1}, \cdots, \mathcal{R}_{i_k,j_k}$ can be denoted by $\mathcal{R} = \mathcal{R}_{i_1,j_1} \cdots \mathcal{R}_{i_k,j_k}$ and by $\mathcal{R}\Gamma$ — the result of consequent reductions of the graph Γ. Note that if the points i and j are not incident then the reduction is trivial and $\mathcal{R}_{i,j}\Gamma = \Gamma$.

Two problems $\mathfrak{A} = (\mathcal{U}, \Gamma, \mathcal{B})$ and $\mathfrak{A}' = (\mathcal{U}', \Gamma', \mathcal{B}')$ are called $\mathcal{R}$-equivalent if there is the sequence of transformation $\mathcal{R} : \mathfrak{A} \to \mathfrak{A}'$, we denote $\mathfrak{A} \overset{\mathcal{R}}{\sim} \mathfrak{A}'$.

We say that the graph without cycles (tree) Γ is *well directed* if it has no non trivial pathes of a length > 1.

Lemma 2. *Let a subgraph $\Gamma|_{\{1,2,\dots,k\}}$ of Γ is a tail with gluing point $k \in \Gamma_0$. Then there is a composition of reductions $\mathcal{R} = \mathcal{R}_{i_1,j_1} \cdots \mathcal{R}_{i_k,j_k}$ with $i_r \in \{1, \dots, k-1\}$, $j_r \in \{1, \dots, k-2\}$ such that: (1) $Q(\Gamma)$ and $Q(\mathcal{R}\Gamma)$ coincides; (2) $\mathcal{R}\Gamma|_{\{1,\dots,k\}}$ is a well directed tail; (3) the direction of an edge $(k-1, k)$ does not change.*

Proof. We proceed by induction on the length of tail k. For $k = 2$ we have nothing to do. On the pictures below the edge is undirected if its direction is not important. So we can apply the assertion of Lemma to the tail $\mathcal{R}\Gamma|_{\{1,\dots,k-1\}}$ with gluing point $k-1 \in \Gamma_0$ and reduce it to the demanded type. Note that by the construction the edges of subgraph $\Gamma|_{\Gamma_0\setminus\{1,\dots,k-1\}}$ do not change. If the obtained graph $\mathcal{R}\Gamma$ is well directed then the proof is over. Otherwise we do the the transformation $\mathcal{R} = \mathcal{R}_{k-2,k-1}\mathcal{R}_{k-2,k-3}$ and obtain:



After that we can apply the assertion of Lemma to the tail $\mathcal{R}\Gamma|_{\{1,\dots,k-2\}}$. □

Corollary 1. *If the graph Γ is Dynkin then there is a composition of reductions $\mathcal{R}$ such that $\mathcal{R}\Gamma$ is a well directed graded graph of the correspondent type. Besides, $Q(\mathcal{R}\Gamma) = Q(\Gamma)$.*

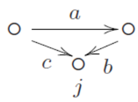
Proof. The case A_n is already proven in Lemma 2. For other Dynkin graphs we use the algorithm from Lemma 2 for the longest tail of the Dynkin graph. After that we can use the same algorithm for other tails depending on direction of the edge of the longest tail that is incident to the branch point. □

Denote by Γ_0^+ (resp., Γ_0^-) the subset of vertices $\mathtt{i} \in \Gamma_0$ such that $\Gamma_1(\mathtt{i},\mathtt{j}) = \varnothing$ (resp., $\Gamma_1(\mathtt{j},\mathtt{i}) = \varnothing$) for any $\mathtt{j} \in \Gamma_0$.

Corollary 2. *Let $(\mathcal{U}, \Gamma, \mathcal{B})$ be the problem from the class Υ, and $\mathtt{j} \in \Gamma_0^+ \cup \Gamma_0^-$. Let $\mathcal{B}_{\Gamma_0 \setminus \{\mathtt{j}\}}$ be a Dynkin forest. Then there exists the composition of reductions $\mathcal{R} : \Gamma \to \Gamma'$ on the subset $\Gamma_0 \setminus \{\mathtt{j}\}$ such that the subgraph $\Gamma'_{\Gamma_0 \setminus \{\mathtt{j}\}}$ is well directed.*

5. Proof of the theorem

Let $(\mathcal{U}, \Gamma, \mathcal{B}) \in \Upsilon$, the quadratic form χ is positive, and Γ is a connected graph. Let $j \in \Gamma_0^{\pm} = \Gamma_0^+ \cup \Gamma_0^-$. Then any edge $a \in \Gamma_1$ is deep on the subgraph $\Gamma_{\Gamma_0 \setminus \{\mathtt{j}\}}$ if and only if it is deep on the whole Γ. Therefore the reductions on $\Gamma_{\Gamma_0 \setminus \{\mathtt{j}\}}$ are correctly defined on Γ. By the induction assumption on the number of points, we can assume that the subgraph $\Gamma_{\Gamma_0 \setminus \{\mathtt{j}\}}$ can be reduced the forrest of Dynkin type. Hence, using Corollary 2, we can assume that all connected components of $\Gamma_{\Gamma_0 \setminus \{\mathtt{j}\}}$ are well directed Dynkin trees. For the further proof, we can assume $j \in \Gamma_0^+$, the case $j \in \Gamma_0^-$ can be considered similarly. Then each contour on Γ is active triangle incident to j of a type



where $\mathtt{d}(c) = ba$, $\deg(c) = \deg(a) + \deg(b) - 1$. Two triangles are incident either to common minimal or common maximal edge.

The point $j \in \Gamma_0$ is called *breaking* if $\Gamma_{\Gamma_0 \setminus \{\mathtt{j}\}}$ has $\geqslant 1$ connected components. For the breaking $j \in \Gamma_0$, let $\Gamma_{\Gamma_0 \setminus \{\mathtt{j}\}} = S_1 \cup \ldots \cup S_q$ be the union of supports for connected components where q denotes the number of components and $|S_1| \geqslant \ldots \geqslant |S_q|$. For the positive form χ the following hold:

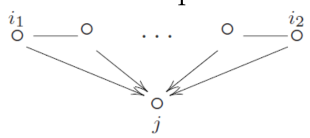
1) q is not more than 3;

2) if $q = 3$ then $|S_3| = 1$, $|S_2| \leqslant 2$, and the subgraph Γ_{S_1} is of A-type;

3) if $q = 2$ then at least one of the subgraphs Γ_{S_1}, Γ_{S_2} is of A-type.

We assume the major component S_1 to be either or not of A-type or the largest (if all components are of A-type).

Let $(\mathcal{U}, \Gamma, \mathcal{B}) \in \Upsilon$, $\mathtt{j} \in \Gamma_0^+ \cup \Gamma_0^-$ and $\Gamma_{\Gamma_0 \setminus \{\mathtt{j}\}}$ be well directed tree. If $\mathtt{i}_1, \mathtt{i}_2 \in \Gamma_{\Gamma_0 \setminus \{\mathtt{j}\}}$ are both incident to $\mathtt{j}$ then all intermediate points are incident to $\mathtt{j}$ too. If $\mathtt{j} \in \Gamma_0^+$ then we have the diagram of a type

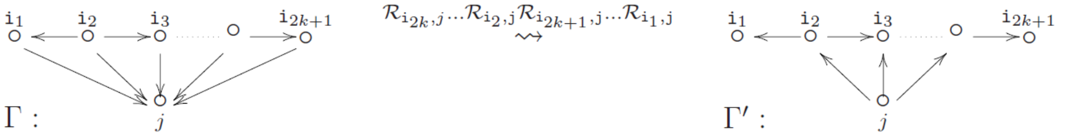


The point $\mathtt{i} \in \Gamma_0$ is called *leaf* if $\Gamma_{\Gamma_0 \setminus \{\mathtt{j},\mathtt{i}\}}$ has the same number of connected components as graph $\Gamma_{\Gamma_0 \setminus \{\mathtt{j}\}}$ or equivalently if $\mathtt{i}$ is incident to only one edge. Assume, some point not incident to $\mathtt{j}$. Then there is a leaf point $\mathtt{i} \in \Gamma_{\Gamma_0 \setminus \{\mathtt{j}\}}$ which is not incident to $\mathtt{j}$.

Lemma 3. *Let $\mathfrak{A} = (\mathcal{U}, \Gamma, \mathcal{B}) \in \Upsilon$, $\chi > 0$. Assume $j \in \Gamma_0^+$ and the subgraph $\Gamma_{\Gamma_0\setminus\{\mathtt{j}\}}$ is a well directed tree of Dynkin type. Then there is the reduction $\mathcal{R}$ such that the obtained problem $\mathcal{R}\mathfrak{A} \in \Upsilon$ has a breaking point from $\Gamma_0\pm$.*

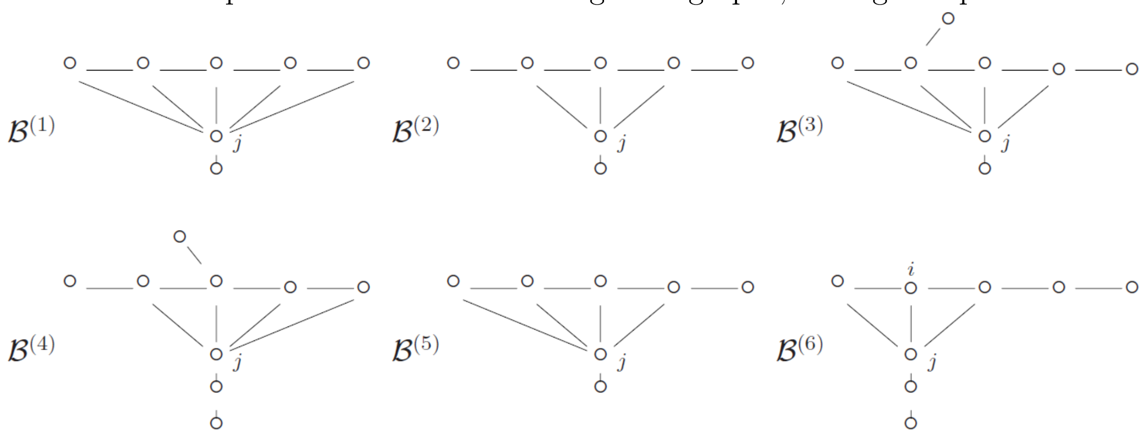
Proof. Assume, there is a leaf point $\mathtt{i}_1 \in \Gamma_{\Gamma_0\setminus\{\mathtt{j}\}}$ which is not incident to $\mathtt{j}$. Since $\Gamma_{\Gamma_0\setminus\{\mathtt{j}\}}$ is a connected graph then $\mathtt{i}_1$ is incident for some $\mathtt{i}_2 \in \Gamma_0 \setminus \{\mathtt{j}\}$. By the construction, $\mathtt{i}_2$ is a breaking point, and it can be transformed to $+$ or $-$ point by the suitable transformation.

It remains to consider the case, all $\mathtt{i} \in \Gamma_{\Gamma_0\setminus\{\mathtt{j}\}}$ are incident to $\mathtt{j}$. The graph $\Gamma_{\Gamma_0\setminus\{\mathtt{j}\}}$ is of A-type because otherwise it has a critical subgraph corresponding $\mathcal{B}^{(1)}$. Therefore, $\Gamma_{\Gamma_0\setminus\{\mathtt{j}\}}$ is a tree graph of A-type. If $\mathtt{i}_1 \in \Gamma_0^\pm$ and $\mathtt{i}_1, \mathtt{i}_2$ are incident, then the edge between $\mathtt{i}_2$, $\mathtt{j}$ are deep, and we can do the transformation $\mathcal{R}_{\mathtt{i}_2,\mathtt{j}}$ to obtain the breaking point $\mathtt{i}_2$. Otherwise, if $\mathtt{i}_1 \notin \Gamma_0^\pm$ and $|\Gamma_0| > 5$ we obtain the following graphs:



Then the graph $\Gamma'' = \mathcal{R}_{\mathtt{j},\mathtt{i}_2}\Gamma'$ has a breaking point $\mathtt{i}_2$ which belongs to $(\Gamma'')_0^-$. □

We exclude the problems with the following subbigraphs, having non positive forms:



Let we give the sketch of the proof of Theorem. Assume $j \in \Gamma_0^+$, the case $j \in \Gamma_0^-$ can be considered similarly. The subgraph $\Gamma_{\Gamma_0\setminus\{\mathtt{j}\}}$ can be reduced to the Dynkin forrest by the induction on the number of points since in this case the transformations on the connected components of $\Gamma_{\Gamma_0\setminus\{\mathtt{j}\}}$ are correctly defined on the whole Γ. Hence, using Corollary 2, we can assume that all components are well directed Dynkin trees. Thereafter, Lemma 3 asserts that each problem to be considered contains a breaking point from Γ_0^+.

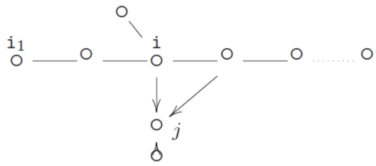
Denote by Υ_0 the subclass of problems $(\mathfrak{A}, \mathtt{j})$ with $\mathfrak{A} \in \Upsilon$ having the breaking point $\mathtt{j} \in \Gamma_0^\pm$ under the conditions that all connected components of $\Gamma_{\Gamma_0\setminus\{\mathtt{j}\}}$ are of A-type. The breaking point $\mathtt{j} \in \Gamma_0^\pm$ is called *hoc breaking point* if $|\Gamma_0 \setminus S_1| \geqslant 3$, hence either or $q = 3$

or the cardinality of second connected component $\geqslant 2$. We say that the hoc point $\mathtt{j}$ is hoc$^+$ (resp., hoc$^-$) point if $\mathtt{j} \in \Gamma_0^+$ (resp., $\mathtt{j} \in \Gamma_0^-$).

Lemma 4. *Let $\mathfrak{A} = (\mathcal{U}, \Gamma, Q) \in \Upsilon$, $\mathtt{j} \in \Gamma_0^+$ be the breaking point. Then there is an equivalent problem $\mathfrak{A}' \overset{\mathcal{R}}{\sim} \mathfrak{A}$ and a breaking point $\mathtt{j}'$ such that $(\mathfrak{A}', \mathtt{j}') \in \Upsilon_0$.*

Proof. If $(\mathfrak{A}, \mathtt{j}) \notin \Upsilon_0$ then, by the induction assumption, the major connected component Γ_{S_1} is a tree having branch point. In this case there are just two connected components due to the positivity of χ.

Firstly we consider the case when Γ has one triangle. If branch point does not incident to this triangle then it can be chose to be a breaking point (possibly after some transformation), and the obtained problem belongs to Υ_0. Otherwise we have a subgraph of a type (probably, without point $\mathtt{i}_1$):

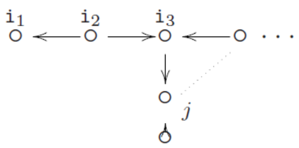


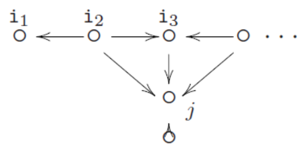
If $\mathtt{i} \in \Gamma_0^-$ then $(\mathfrak{A}, \mathtt{i}) \in \Upsilon_0$. Otherwise $\mathtt{i}$ is +point on $\Gamma_{\Gamma_0 \setminus \{\mathtt{j}\}}$ and $\mathtt{i}$ is a hot$^+$ breaking point for $\mathcal{R}_{\mathtt{j},\mathtt{i}}$, besides $(\mathcal{R}_{\mathtt{j},\mathtt{i}}, \mathtt{i}) \in \Upsilon_0$. Then, excluding the critical problem with bigraph $\mathcal{B}^{(2)}$, we conclude that the problem has two or three clear triangles.

Now we show that there is an equivalent problem $\mathfrak{A}' \overset{\mathcal{R}}{\sim} \mathfrak{A}$ which either or belongs to Υ_0 or has a hoc breaking point from $\Gamma_0^{\pm}$. So we assume $|\Gamma_0 \setminus S_1| = 2$.

Let $\mathtt{i}_1, \mathtt{i}_2, \mathtt{i}_3 \in S_1$, $\Gamma_{\mathtt{i}_1,\mathtt{i}_2,\mathtt{i}_3}$ be connected graph and $\mathtt{i}_1$ be a leaf point. If $\mathtt{i}_l$ is not incident to $\mathtt{j}$, $l = 1, 2, 3$, then there is a hoc breaking point $\mathtt{k} \in \{\mathtt{i}_1, \mathtt{i}_2, \mathtt{i}_3\} \cap \Gamma_0^{\pm}$.

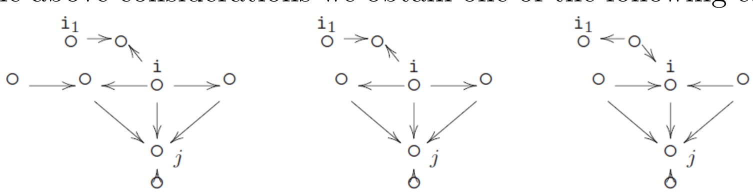
We consider also the cases: 1) $\mathtt{i}_1, \mathtt{i}_2$ both do not branch and do not incident to $\mathtt{j}$, and $\mathtt{i}_3$ is incident to $\mathtt{j}$; and 2) $\mathtt{i}_1 \in S_1$ be a leaf point, $\mathtt{i}_2 \in S_1 \cap \Gamma_0^-$ be incident to $\mathtt{i}_1$ and $\mathtt{i}_2$ do not a branch point. For the first case, if $\mathtt{i}_3 \in \Gamma_0^-$ then it is a hoc$^-$ breaking point. We have one of the following problems:

1) 

2) 

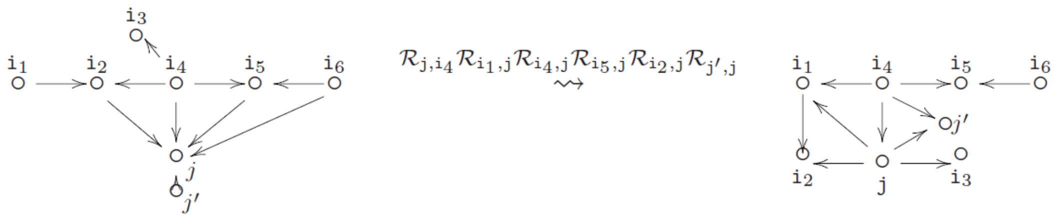
For the both cases, the problem $\mathcal{R}_{\mathtt{j},\mathtt{i}_3}\mathfrak{A} \in \Upsilon$ has a hoc$^+$ point $\mathtt{i}_3$.

Consider the case when Γ has two clear triangles. Taking into account the critical bigraph $\mathcal{B}^{(2)}$ and the above considerations we obtain one of the following cases (probably, without point $\mathtt{i}_1$):



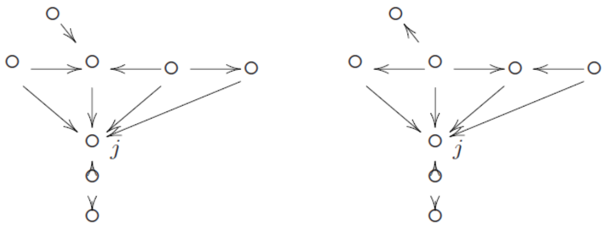
. For all cases, the graphs can be reduced to tree graph directly.

If Γ has three clear triangles, then we obtain taking into account the critical bigraph $\mathcal{B}^{(2)}$ and the above considerations we obtain the following case (probably, without $\mathtt{i}_1$):



Then $\mathtt{i}_4$ is a hoc$^-$ point. Therefore, it remains to consider the case when $\mathfrak{A}$ has a hoc point and three clear triangles.

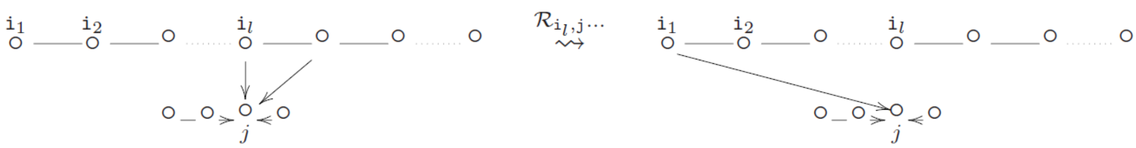
We exclude the bigraphs $\mathcal{B}^{(4)}$, having non positive quadratic forms. Then it remains to consider the following cases:



It is simply to verify that both the cases are reduced directly to the Dynkin tree. □

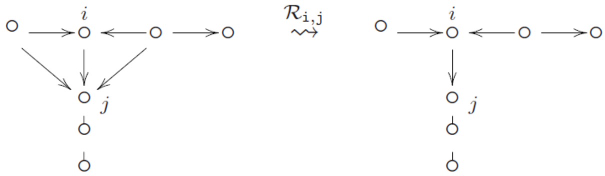
Lemma 5. *Let* $\mathtt{j} \in \Gamma_0^+$ *be the breaking point and* $(\mathfrak{A}, \mathtt{j}) \in \Upsilon_0$. *Then there is an equivalent problem* $\mathfrak{A}' \overset{\mathcal{R}}{\sim} \mathfrak{A}$ *such that* $\mathfrak{A}'$ *is a Dynkin tree.*

Proof. If there are three connected components, then $|S_3| = 1$, $|S_2| = 1$ or $|S_2| = 2$. If there are more then one triangle contour on $S_1 \cup \mathtt{j}$ then problem does not have positive quadratic form. There can be only one triangle, which can be moved to the leaf point of the connected component using the reduction of one of the deep edges and finally we obtain the Dynkin tree:



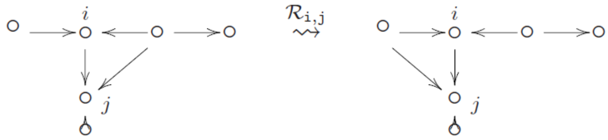
Note that due to positivity of χ if $|S_2| = 2$ then $|S_1| \leqslant 4$ and we obtain the E-reduced problem, if $|S_2| = 1$ then $|S_1|$ can be arbitrary and we obtain the D reduced problem.

Consider the case of two connected components. It has two subcases: $|S_2| = 1$ and $|S_1| \geq 2$. Consider the subcase $|S_1| \geq 2$. If there is only one triangle then the problem can be reduced to the tree of A-type using the same reductions as in case above (with three components). The problem with two triangles is critical of type $\mathcal{B}^{(2)}$ or $\mathcal{B}^{(6)}$ or is equivalent to problem E_7 if $|S_1| = 4$:



The case with opposite direction is more complicated, but is also reduced to E_7.

Consider the subcase $|S_2| = 1$. We will try to obtain the equivalent problem with $|S_1| \geq 2$. If the component S_1 has a tail of the length 2 and more it is obvious. We consider the problem with triangle containing end point of S_1. It can have one or two triangles. Those cases are reduced directly to Dynkin problem. The problem with three triangles is critical of type $\mathcal{B}^{(5)}$. The last case is the problem for which S_1 has two tails each containing one point. It can have only one triangle, then after the reduction



we obtain the problem with $|S_1| \geq 2$. The problem with two triangles is critical $\mathcal{B}^{(6)}$. $\square$

The proof of Theorem 1 simply follows from Lemmas 3, 4, 5.

Note that using the transformation of raising the degree for dgc $\mathcal{U}$ we can obtain the finale graphs of Dynkin type having only the even degrees of edges. In another words, the obtained bigraph is a 0-forrest of Dynkin type.

References

[1] *Drozd Y. A.* Reduction algorithm and representations of boxes and algebras // Comtes Rendue Math. Acad. Sci. Canada. — 2001. — 23. — P. 97–125.

[2] *Roiter A. V.* The roots of integral quadratic forms // Quadratic forms and representations. — 1978. — P. 3–17.

[3] Ovsienko S. A. *Integer weakly positive forms* // Shurian Matrix problems and quadratic forms. — 1978. — P. 3–17.

[4] *Barot M., de la Pena J. A.* The Dynkin type of a non-negative unit form // Expo. Math. — 1999. — **17**. — P. 339–348.

[5] *Gnatiuk O., Golovashchuk N.* Differential graded categories with positive quadratic form // Visnyk Kyiv National Taras Schevchenko University. — 2012. — 3. — P. 15–19.