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On centralizers of elements in the Lie algebra $W_2(K)$

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Abstract. Let K be a field of characteristic zero and $K[x, y]$ the polynomial ring. Denote by $W_2(K)$ the Lie algebra of all K-derivations of $K[x, y]$. Centralizers of elements and maximal abelian subalgebras of the algebra $W_2(K)$ are studied. The structure of a centralizer $C_{W_2}(D)$ depends on the $kerD$ in the field of rational functions $K(x, y)$ (the derivation D can be naturally extended on $K(x, y)$). In particular, if $kerD$ in $K(x, y)$ coincides with K or does not contain nonconstant polynomials, then $C_{W_2}(D)$ is of finite dimension over K.

Introduction

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero and $\mathbb{K}[x, y]$ be the polynomial algebra over $\mathbb{K}$. The Lie algebra $W_2 = W_2(\mathbb{K})$ of all $\mathbb{K}$-derivations of $\mathbb{K}[x, y]$ was studied by many authors (its finite dimensional subalgebras were described by S. Lie in case $\mathbb{K} = \mathbb{C}$ [**2**], see also [**3**], [**4**]). The structure of subalgebras of the Lie algebra $W_2(\mathbb{K})$ is of great interest because its elements can be considered as vector fields on $\mathbb{K}^2$ with polynomial coefficients. In this paper, we give a characterization of centralizers of elements in the Lie algebra $W_2(\mathbb{K})$. Given a derivation $D \in W_2(\mathbb{K})$ one can consider its extension on the field $\mathbb{K}(x, y)$ of rational functions by the rule $D(\frac{f}{g}) = \frac{D(f)g - fD(g)}{g^2}$ and the subfield of constants $\ker D$. We prove (Theorem 1) that $C_{W_2}(D)$ is one-dimensional or two-dimensional (over $\mathbb{K}$) in case $\ker D = \mathbb{K}$. If $\ker D$ contains a nonconstant polynomial then $C_{W_2}(D)$ is infinite dimensional, its structure is described. The last case when $\ker D$ contains a rational function, but not a nonconstant polynomial yields against finite dimensional centralizers but of more complicated structure. Using these results we give a characterization of maximal abelian subalgebras of $W_2(\mathbb{K})$ (Theorem 2).

We use standard notations, the ground field $\mathbb{K}$ is algebraically closed of characteristic zero. A nonzero derivation D will be called reduced if from any equality $D = hD_1, h \in \mathbb{K}[x, y], D_1 \in W_2(\mathbb{K})$ it follows that $h \in \mathbb{K}^\star$. It is obvious that every derivation $D \in W_2(\mathbb{K})$ can be written in the form $D = hD_0$ where $h \in \mathbb{K}[x, y]$ and D_0 is reduced. The field $\mathbb{K}(x, y)$ will be denoted by R.

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Every polynomial $p \in \mathbb{K}[x, y]$ defines a derivation $D_p \in W_2(\mathbb{K})$ (it is called a Jacobi derivation) by the rule: $D_p(h) = detJ(p, h)$, where $J(p, h)$ is the Jacobi matrix of the polynomials p, h If p is irreducible we denote by δ_p a reduced derivation corresponding to D_p. Analogously every irreducible fraction $p/q \in \mathbb{K}(x, y)$ defines a Jacobi derivation $D_{p,q} = qD_p - pD_q \in W_2(\mathbb{K})$, a reduced derivation corresponding to the last one will be denoted by $\delta_{p,q}$.

1. On centralizers of elements in $W_2(\mathbb{K})$

Let D be a nonzero derivation of $\mathbb{K}[x, y]$. It is obvious that D satisfies one of the following conditions: (1) $\ker D$ in $\mathbb{K}(x, y)$ coincides with $\mathbb{K}$. (2) $\ker D$ in $\mathbb{K}(x, y)$ contains a nonconstant polynomial. (3) $\ker D$ in $\mathbb{K}(x, y)$ is different from $\mathbb{K}$ and does not contain any nonconstant polynomial. We consider these cases in Lemmas 2, 4 and 5.

The next statement can be directly proved.

Lemma 1. *Let $\varphi(t) = u(t)/v(t) \in \mathbb{K}(t)$ be an irreducible fraction,* $\deg u = m, \deg v = n$. *If the polynomials $p, q \in \mathbb{K}[x, y]$ are algebraically independent over $\mathbb{K}$ and irreducible, then the rational function $\varphi(p/q)$ from $\mathbb{K}(x, y)$ can be written in the form $\varphi(p/q) = \overline{u}/\overline{v}$, where $\overline{u}, \overline{v}$ are homogeneous polynomials in p, q of the same degree* $\max(m, n)$. *These polynomials can be chosen to be coprime.*

Lemma 2. *Let $D \in W_2(\mathbb{K})$ be a derivation such that* $\ker D$ *in $\mathbb{K}(x, y)$ is equal to $\mathbb{K}$. Then the centralizer $C = C_{W_2}(D)$ coincides with $\mathbb{K}D$, or $C = \mathbb{K}D + \mathbb{K}D_1$ for a derivation D_1 such that D, D_1 are linearly independent over $\mathbb{K}(x, y)$.*

Proof. Let first $C = C_{W_2}(D)$ have rank 1 over $\mathbb{K}(x, y)$. If $\dim_{\mathbb{K}} C = 1$, then $C = \mathbb{K}D$ and all is done. Let $\dim_{\mathbb{K}} C > 1$ and D_1 be such an element of C that D, D_1 are linearly independent over $\mathbb{K}$. Since $\mathrm{rk}_R C = 1$, there exist polynomials $f, g \in \mathbb{K}[x, y]$ such that $fD + gD_1 = 0$ (at least one of the polynomials is nonconstant). Then there exist a reduced derivation D_0 and polynomials $\alpha, \beta \in \mathbb{K}[x, y]$ such that $D = \alpha D_0, D_1 = \beta D_0$. Since D, D_1 are linearly independent over $\mathbb{K}$, at least one of the polynomials α, β is nonconstant, and α/β is a nonconstant rational function. Further,

$$0 = [D, D_1] = [\alpha D_0, \beta D_0] = (\alpha D_0(\beta) - D_0(\alpha)\beta)D$$

and therefore $\alpha D_0(\beta) - D_0(\alpha)\beta = 0$. But then $D(\frac{\alpha}{\beta}) = 0$ and since $\frac{\alpha}{\beta} \notin \mathbb{K}$ we get a contradiction to our assumption on D. Therefore $\dim_{\mathbb{K}} C = 1$ and $C = C_{W_2}(D) = \mathbb{K}D$.

Let now C be of rank 2 over $R = \mathbb{K}(x, y)$. Then there exists an element $D_1 \in C$ such that D_1, D are linearly independent over $\mathbb{K}(x, y)$. Let D_2 be an arbitrary element of C, write $D_2 = uD + vD_1$ for some $u, v \in \mathbb{K}(x, y)$. We have

$$[D, D_2] = D(u)D + D(v)D_1 = 0,$$

and taking into account the linear independence of D, D_1 over $\mathbb{K}(x,y)$ we get $D(u)=0$ and $D(v)=0$. Since $\ker D=\mathbb{K}$ (in $\mathbb{K}(x,y)$) we see that $u,v\in\mathbb{K}$ and $D_2\in\mathbb{K}D+\mathbb{K}D_1$ i.e $C\subseteq\mathbb{K}D+\mathbb{K}D_1$. As it holds obviously $\mathbb{K}D+\mathbb{K}D_1\subseteq C$ we obtain $C=\mathbb{K}D+\mathbb{K}D_1$. □

Lemma 3. *([**1**]) Let $D\in W_2(\mathbb{K})$ be a derivation such that $\ker D$ in $\mathbb{K}(x,y)$ contains a nonconstant polynomial. Then $D=h\delta_p$, where $h\in\mathbb{K}[x,y], \delta_p$ is a reduced derivation corresponding to the Jacobi derivation D_p for an irreducible polynomial $p=p(x,y)\in\mathbb{K}[x,y]$.*

Definition 1. (see also [**1**]). (1) Let $p=p(x,y)\in\mathbb{K}[x,y]$ be an irreducible polynomial. A polynomial $f=f(x,y)$ will be called p-free if f is not divisible by any polynomial in p of positive degree. Every polynomial $g\in\mathbb{K}[x,y]$ can be written in the form $g=g_0g_1$, where g_0 is a p-free polynomial and $g_1=g_1(p)$ is a polynomial of p (may be $g_1=const$). The degree of the polynomial $g_1(p)$ in p will be called the p-degree of g and denoted by $\deg_p g$.

(2) Let p and q be algebraically independent irreducible polynomials from the ring $\mathbb{K}[x,y]$. A polynomial $f(x,y)\in\mathbb{K}[x,y]$ will be called p-q-free if f is not divisible by any homogeneous polynomial in p and q of positive degree. As earlier one can write every polynomial $g\in\mathbb{K}[x,y]$ in the form g_0g_1, where g_0 is a p-q-free polynomial and $g_1=h(p,q)$ for some homogeneous polynomial $h(s,t)\in\mathbb{K}[s,t]$. The (total) degree of h in s,t will be called the p-q-degree of g and denoted by $\deg_{p-q} g$.

Lemma 4. *Let $D\in W_2(\mathbb{K})$ be a derivation such that $\ker D$ in $\mathbb{K}(x,y)$ contains a nonconstant polynomial. Then $D=hf(p)\delta_p$ where p is an irreducible polynomial from $\ker D$, h is a p-free polynomial and δ_p is a reduced derivation corresponding to the Jacobi derivation D_p. The centralizer $C_{W_2}(D)$ is one of the following algebras:*

(1) $C_{W_2}(D)=\mathbb{K}[p]h\delta_p$

(2) $C_{W_2}(D)=\mathbb{K}[p]h\delta_p+\mathbb{K}[p]D_1$

for some $D_1\in C_{W_2}(D)$ such that D_1, D are linearly independent over $\mathbb{K}(x,y)$.

Proof. Using Lemma 3 we can write the derivation D in the form $D=g\delta_p$, where δ_p is a reduced derivation corresponding to D_p. We write the polynomial $g=g(x,y)$ in the form $g=hf(p)$, where $f(p)$ is a polynomial of p (may be $f(p)=const$) and $h=h(x,y)$ is a p-free polynomial. Let first $\mathrm{rk}_R C=1$. Then any derivation $D_1\in C_{W_2}(D)$ is of the form $D_1=g_1\delta_p$, for some polynomial $g_1\in\mathbb{K}[x,y]$. It follows from the equality

$$0=[D,D_1]=[hf(p)\delta_p,g_1\delta_p]=(f(p)\delta_p(h)g-hf(p)\delta_p(g_1))\delta_p$$

that $\delta_p(h)g_1-h\delta_p(g_1)=0$ and therefore $\delta_p(g_1/h)=0$. But then $g_1/h=\varphi(p)$ for some rational function $\varphi(t)\in\mathbb{K}(t)$ (since $\ker\delta_p=\mathbb{K}(p)$ in $\mathbb{K}(x,y)$). Thus $g_1/h=u(p)/v(p)$ for some polynomials $u(t),v(t)\in\mathbb{K}[t]$ (these polynomials can be chosen coprime). From the last equality we get $g_1 v(p) = hu(p)$. If the polynomial $v(t)$ is nonconstant, then taking into account the condition $\mathbb{K} = \overline{\mathbb{K}}$ we see that every divisor of $v(p)$ of the form $p - \lambda_i, \lambda_i \in \mathbb{K}$ divides the polynomial h. But the latter is impossible because of choice of h and therefore $v(p) = const$. Denote $u_1(t) = u(t)/v \in \mathbb{K}[t]$. We get $g_1 = hu_1(p)$ and $D_1 = g_1\delta_p = hu_1(p)\delta_p \in \mathbb{K}[p]h\delta_p$. Since the element D_1 was chosen arbitrarily we obtain $C_{W_2}(D) \subseteq \mathbb{K}[p]h\delta_p$. It is obvious that $\mathbb{K}[p]h\delta_p \subseteq C_{W_2}(D)$ and therefore $C_{W_2}(D) = \mathbb{K}[p]h\delta_p$. The centralizer $C_{W_2}(D)$ is of type 1.

Let now $C_{W_2}(D)$ be of rank 2 over $R = \mathbb{K}(x, y)$. Note that for any derivation $D_1 \in C_{W_2}(D)$ satisfying the condition $DD_1 = D_1 D$ it follows that $D_1(p) \in \ker D$. Since $\ker D = \mathbb{K}[p]$ (in $\mathbb{K}(x, y)$), then $D_1(p) = f_1(p)$ for some polynomial $f_1(t) \in \mathbb{K}[t]$. Take an element $D_1 \in C_{W_2}(D)$ such that the polynomial $f_1(t)$ has the possible lowest degree. Take now any derivation $D_2 \in C_{W_2}(D)$ and let $D_2(p) = f_2(p)$. Let us show that the polynomial $f_2(t)$ is divisible by $f_1(t)$. Really, let this is not the case and for some $f_2(t) \in \mathbb{K}[t]$ the greatest common divisor $gcd(f_2, f_1) = g(t)$ be of degree less than $min(\deg f_1, \deg f_2)$. Then there exist polynomials $a(t), b(t) \in \mathbb{K}[t]$ such that $g(t) = a(t)f_1(t) + b(t)f_2(t)$. The derivation $D_3 = a(p)D_1 + b(p)D_2$ satisfies obviously the condition $[D, D_3] = 0$ and $D_3(p) = g(p)$. The latter contradicts to the choice of the derivation D_1. Therefore $f_2(t)$ is divisible by $f_1(t)$. Set $\mu(t) = f_2(t)/f_1(t)$. It is easy to see that

$$(D_2 - \mu(p)D_1)(p) = D_2(p) - \mu(p)D_1(p) = f_2(p) - \mu(p)f_1(p) = 0.$$

The latter means in view of Lemma 1 that $D_2 - \mu(p)D_1 = g\delta_p$ for some polynomial $g \in \mathbb{K}[x, y]$. Since $D_2 - \mu(p)D_1 \in C$, it follows from above proven that $g = u(p)h$ for some $u(p) \in \mathbb{K}[p]$. Thus, $D_2 \in \mathbb{K}[p]D_1 + \mathbb{K}[p]h\delta_p$. The inverse inclusion holds obviously and therefore $C_{W_2}(D) = \mathbb{K}[p]D_1 + \mathbb{K}[p]h\delta_p$. □

Lemma 5. *Let $D \in W_2(\mathbb{K})$ be such a derivation that $\ker D \neq \mathbb{K}$ in $\mathbb{K}(x, y)$ and $\ker D$ does not contain any nonconstant polynomial. Then $D = hf(p, q)\delta_{p,q}$, where p, q are algebraically independent over $\mathbb{K}$ irreducible polynomials such that $\ker D = \mathbb{K}(\frac{p}{q})$, $f(p, q)$ is a homogeneous polynomial in p and q of degree $m \geq 0$, the polynomial h is p-q-free and $\delta_{p,q}$ is a reduced derivation corresponding to $qD_p - pD_q$. The centralizer $C = C_{W_2}(D)$ is one of the following algebras:*

(1) $C = \mathbb{K}[p, q]_m h\delta_{p,q}$, where $\mathbb{K}[p, q]_m$ is the space of all homogeneous polynomials in p, q of degree $m = \deg_{p-q} f$, in particular $\dim_{\mathbb{K}} C = m + 1$.

(2) $C = (\mathbb{K}(\frac{p}{q})D + \mathbb{K}(\frac{p}{q})D_1) \cap W_2(\mathbb{K})$, where $D_1 \in C$, such that D, D_1 are linearly independent over $\mathbb{K}(x, y)$. The subalgebra C is finite dimensional over $\mathbb{K}$, and if $D = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$, $D_1 = P_1\frac{\partial}{\partial x} + Q_1\frac{\partial}{\partial y}$, $\Delta = PQ_1 - P_1 Q$, and $\deg_{p-q}\Delta = s$, then $\dim_{\mathbb{K}} C \leq m + s + 2$, where $m = \deg_{p-q} f$.

Proof. Since $\ker D \neq \mathbb{K}$, the subfield $\ker D$ contains a nonconstant rational function. Note that $\ker D$ in $\mathbb{K}(x, y)$ is algebraically closed in $\mathbb{K}(x, y)$, so $\mathrm{tr.deg}_{\mathbb{K}} \ker D = 1$. The Gordan's Theorem (see [**6**], Th. 3) yields now that $\ker D = \mathbb{K}(\frac{p}{q})$ for a nonconstant rational function p/q. The polynomials p and q can be chosen to be irreducible (see, for example [**5**]). It can be easily shown that $D = hf(p, q)\delta_{p,q}$ where $\delta_{p,q}$ is a reduced derivation correspond to $qD_p - pD_q$, h is p-q-free and $f(p, q)$ is a homogeneous polynomial in p, q. Set $m = \deg_{p-q} f$.

Let first $C = C_{W_2}(D)$ be of rank 1 over $\mathbb{K}(x, y)$. Take any element $D_1 \in C$. Then $D_1 = d_1\delta_{p,q}$ for some polynomial d_1, the polynomial d_1 can be written in the form $d_1 = f_1h_1$, where $f_1 = f_1(p, q)$ is a homogeneous polynomial in p, q and h_1 is p-q-free. The derivations D and D_1 satisfy the condition

$$[D, D_1] = [hf(p, q)\delta_{p,q}, h_1f_1\delta_{p,q}] = 0.$$

But then $\delta_{p,q}(hf)h_1f_1 - hf\delta_{p,q}(h_1f_1) = 0$ and therefore $\delta_{p,q}(hf/h_1f_1) = 0$. The latter means that $\frac{hf}{h_1f_1} \in \ker D = \mathbb{K}(\frac{p}{q})$ and $hf/h_1f_1 = u(p, q)/v(p, q)$ for some homogeneous (in p, q) polynomials u, v of the same degree (see Lemma 1). We can choose these polynomials to be coprime as polynomials in p, q. But then they are coprime as polynomials in x, y because $p - \lambda_i q$ and $p - \lambda_j q$ are coprime provided that $\lambda_i \neq \lambda_j$.

It follows from these considerations that $hfv = h_1f_1u$ with homogeneous polynomials fv, f_1u in p, q and p-q-free polynomials h, h_1. Since the decomposition into product of a p-q-free polynomial and a homogeneous in p, q polynomial is unique up to nonzero scalar multiple it follows that $h_1 = hc$ for some $c \in \mathbb{K}^\star$. As $\deg_{p-q} u = \deg_{p-q} v$ by the choice of these polynomials we see that $\deg_{p-q} f_1 = \deg_{p-q} f = m$. Then $D_1 = f_1h_1\delta_{p,q} \in \mathbb{K}[p, q]_m h\delta_{p,q}$ where $\mathbb{K}[p, q]_m$ is the vector space of all homogeneous polynomials in p, q of degree m in p, q. One can easily show that $\mathbb{K}[p, q]_m h\delta_{p,q} \subseteq C$ and therefore $C = \mathbb{K}[p, q]_m h\delta_{p,q}$. The centralizer is of type 1 of Lemma.

Let now the rank $C_{W_2}(D)$ be equal to 2 (over $\mathbb{K}(x, y)$). Write $D = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$, $D_1 = P_1\frac{\partial}{\partial x} + Q_1\frac{\partial}{\partial y}$ with $P, Q, P_1, Q_1 \in \mathbb{K}[x, y]$ and let $\Delta_1 = PQ_1 - P_1Q$. Further, take an another element $D_2 \in C$ such that D, D_2 are also linearly independent over $\mathbb{K}(x, y)$. Then $D_2 = \alpha D + \beta D_1$ for some $\alpha, \beta \in \mathbb{K}(x, y)$. It follows from the relations $0 = [D, D_2] = [D, \alpha D + \beta D_1] = D(\alpha)D + D(\beta)D_1$ that $D(\alpha) = D(\beta) = 0$ (because of linearly independence of D, D_1). But then $\alpha, \beta \in \mathbb{K}(\frac{p}{q})$ (recall that $\ker D$ in $\mathbb{K}(x, y)$ coincides with $\mathbb{K}(\frac{p}{q})$) and therefore $C \subseteq (\mathbb{K}(\frac{p}{q})D + K(\frac{p}{q})D_1) \cap W_2(\mathbb{K})$. The inverse to this inclusion also holds, so we have $C = (\mathbb{K}(\frac{p}{q})D + K(\frac{p}{q})D_1) \cap W_2(\mathbb{K})$.

Write now the derivation D_2 in the form $D_2 = P_2\frac{\partial}{\partial x} + Q_2\frac{\partial}{\partial y}$ with $P_2, Q_2 \in \mathbb{K}[x, y]$ and denote $\Delta_2 = PQ_2 - P_2Q$. Since $P_2 = \alpha P + \beta P_1$ and $Q_2 = \alpha Q + \beta Q_1$, we have $\Delta_2 = \beta\Delta_1$. The rational function $\beta \in \mathbb{K}(p/q)$ can be written in the form $\beta = u/v$, where u, v are homogeneous polynomials in p, q and $\deg_{p-q} u = \deg_{p-q} v$ (see Lemma 1). Then we obtain from the equality $\Delta_2 = \beta\Delta_1$ and condition $\Delta_1, \Delta_2 \in \mathbb{K}[x, y]$ that the polynomials Δ_1 and Δ_2 have the same p-q-degree. Besides, these polynomials have the same p-q-free part up to nonzero scalar multipliers. Let $\deg_{p-q}\Delta_1 = s$. Note that the vector space $\mathbb{K}[p, q]_s$ of all homogeneous polynomials of degree s in p, q has dimension $s + 1$ over $\mathbb{K}$. The centralizer $C = C_{W_2}(D)$ has a $\mathbb{K}$-subspace C_0 consisting of all derivations linearly dependent with D. By the above proven the subspace C_0 is of dimension m over $\mathbb{K}$ where m is the p-q-degree of the polynomial f from the decomposition $D = hf\delta_{p,q}$. Take arbitrary derivations $T_1, \dots, T_{s+2}$ from C, write down $T_i = P_i\frac{\partial}{\partial x} + Q_i\frac{\partial}{\partial y}$, $i = 1, \dots, s+2$, and denote $\Delta_i = PQ_i - P_iQ$. Since the determinantes Δ_i have the same p-q-free part (up to nonzero scalar multipliers) and $\dim_{\mathbb{K}} \mathbb{K}[p, q]_s = s + 1$, there exist elements $c_1, \dots, c_{s+2} \in \mathbb{K}$ such that $c_1\Delta_1 + \dots + c_{s+2}\Delta_{s+2} = 0$ and at least one of c_i is nonzero. Consider the derivation $T = c_1T_1 + \dots + c_{s+2}T_{s+2} = U\frac{\partial}{\partial x} + V\frac{\partial}{\partial y}, U, V \in \mathbb{K}[x, y]$ from the centralizer C. It is obvious that $PV - QU = 0$ and this equality implies that D and T are linearly dependent over $\mathbb{K}(x, y)$, i.e. $T \in C_0$. Therefore $\dim C/C_0 \leq s + 1$. But then the dimension of C over $\mathbb{K}$ does not exceed $(m + 1) + (s + 1) = m + s + 2$. □

Theorem 1. *Let D be an arbitrary nonzero element of $W_2(\mathbb{K})$. Then the centralizer $C = C_{W_2}(D)$ is a subalgebra of one of the following types:*

(1) $C = \mathbb{K}D$, if $\ker D$ in $\mathbb{K}(x, y)$ coincides with $\mathbb{K}$.

(2) $C = \mathbb{K}D + \mathbb{K}D_1$, if $\ker D$ in $\mathbb{K}(x, y)$ coincides with $\mathbb{K}$ and there exists D_1 such that $[D, D_1] = 0$ and D, D_1 are linearly independent over $\mathbb{K}(x, y)$.

(3) $C = \mathbb{K}[p]h\delta_p$, if $\ker D$ in $\mathbb{K}(x, y)$ contains a nonconstant polynomial, this polynomial p can be chosen irreducible, $D = hf\delta_p$, where f is a polynomial in p, h is p-free and δ_p is a reduced derivation corresponding to D_p.

(4) $C = \mathbb{K}[p, q]_m h\delta_{p,q}$, if $\ker D$ contains a nonconstant rational function p/q and does not contain any nonconstant polynomial, $\ker D = \mathbb{K}(\frac{p}{q})$, $D = hf\delta_{p,q}$, where f is a homogeneous polynomial in $,q$ of degree m, h is a p-q-free polynomial and $\delta_{p,q}$ is a reduced derivation corresponding to $qD_p - pD_q$.

(5) $C = (\mathbb{K}(\frac{p}{q})D + \mathbb{K}(\frac{p}{q})D_1) \cap W_2(\mathbb{K})$, where D satisfies all the conditions of the previous part of Theorem, D and D_1 are linearly independent over $\mathbb{K}(x, y)$ $[D_1, D] = 0$. If $D = P\frac{\partial}{\partial x} + Q\frac{\partial}{\partial y}$, $D_1 = P_1\frac{\partial}{\partial x} + Q_1\frac{\partial}{\partial y}$, and $\Delta = PQ_1 - P_1Q$ then $\dim_{\mathbb{K}} C \leq m + s + 2$, where m as in part 4 of Theorem and $s = \deg_{p-q}\Delta$.

Proof. See Lemmas 2, 4 and 5. □

Corollary 1. *If $D \in W_2(\mathbb{K})$ and $C_{W_2}(D)$ is infinite dimensional over $\mathbb{K}$, then $\ker D$ contains a nonconstant polynomial which can be chosen to be irreducible.*

Theorem 2. *Let L be a maximal abelian subalgebra of the Lie algebra $W_2(\mathbb{K})$. Then L is one of the following algebras:*

(1) One-dimensional of the form $\mathbb{K}D$ *where* $D \in W_2(\mathbb{K}$ *and* $\ker D$ *in* $\mathbb{K}(x,y)$ *coincides with* $\mathbb{K}$.

(2) Two-dimensional of the form $\mathbb{K}D + \mathbb{K}D_1$ *where* D, D_1 *are linearly independent over* $\mathbb{K}(x,y)$.

(3) Finite dimensional of the form $\mathbb{K}[p,q]_m h\delta_{p,q}$, *where* $h \in \mathbb{K}[x,y]$, $\mathbb{K}[p,q]_m$ *is the vector space of all homogeneous in* p, q *polynomials of degree* m *(see Theorem* 1*).*

(4) Infinite dimensional of the form $\mathbb{K}[p]h\delta_p$, *where* $h \in \mathbb{K}[x,y]$, $\mathbb{K}[p]$ *is the vector space of polynomials in* p *(see Theorem* 1*).*

Proof. Let L be a maximal abelian subalgebra of $W_2(\mathbb{K})$. If $\mathrm{rk}_{\mathbb{K}(x,y)} L = 2$ then L contains elements D_1, D_2 which form a basis of $W_2(\mathbb{K})$ over $\mathbb{K}(x,y)$ (as a vector space). But then every element D of L can be written in the form $D = \alpha_1 D_1 + \alpha_2 D_2$ for some $\alpha_1, \alpha_2 \in \mathbb{K}(x,y)$. Since $[D, D_1] = [D, D_2] = 0$ we have that $D_1(\alpha_i) = 0, \quad D_2(\alpha_i) = 0, i = 1, 2$. The latter means that $\alpha_1, \alpha_2 \in \mathbb{K}$ and therefore $L = \mathbb{K}D_1 + \mathbb{K}D_2$. The Lie algebra L is of type 2 of this Theorem.

Let now $\mathrm{rk}_{\mathbb{K}(x,y)} L = 1$. Take any nonzero element $D \in L$. If $\dim_{\mathbb{K}} L = \infty$, then $L \subseteq C_{W_2}(D)$ and $C_{W_2}(D) = \mathbb{K}[p]h\delta_p$ by Theorem 1. Since $C_{W_2}(D)$ is abelian we see that $L = C_{W_2}(D)$ and L is of type 4. If $\dim_{\mathbb{K}} L < \infty$ then one can analogously show that L is of type 1 or 3. □

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