Науковий часопис НПУ імені М. П. Драгоманова. Серія 1. Фізико-математичні науки
Київ: НПУ імені М. П. Драгоманова. — 2013, № 14.— С. 10–13.

Primitive graded groups with the minimal condition for non-normal subgroups

N. S. Chernikov

Department of Algebra, Institute of Mathematics,

National Academy of Sciences of Ukraine, Kyiv, Ukraine

Анотація. Автор доводить, зокрема, що вказані групи, — це точно черніковські групи та дедекіндові групи.

Abstract. The author proves, in particular, that the mentioned groups are exactly the Chernikov groups and the Dedekind groups.

To the memory of my father Professor Sergei N. Chernikov, who was an outstanding mathematician and the founder of the well-known large and powerful algebraic school

Let $min - \bar{n}$ be the minimal condition for non-normal subgroups. Obviously, the class of groups with $min - \bar{n}$ is closed with respect to forming subgroups and factor groups.

Remind that a group with the minimal condition for (all) subgroups is called Artinian. (Chernikov groups present some special case of Artinian groups.)

Remind that a group with all subgroups normal is called Dedekind.

Clearly, Artinian groups and Dedekind groups satisfy $min - \bar{n}$. The remarkable S. N. Chernikov's Theorem [**1**] asserts: *A group, having a series with finite factors, (in particular, an RN-group, a locally solvable group) satisfies $min - \bar{n}$ iff it is Chernikov or Dedekind.*

Remind that the Dedekind groups are completely described by famous R. Baer's Theorem [**2**].

In the present article, the author continues to investigate the groups with $min - \bar{n}$.

Introduce the definition.

E-mail: chern@imath.kiev.ua

Definition 1. The group G will be called primitive graded, if for every g, $h \in G$, the subgroup $< g, g^h >$ possesses a subgroup of finite index $\neq 1$ whenever g is a p-element $\neq 1$ with some odd p and also $[g^p, h] = 1$ and the subgroup $< g, h >$ is periodic.

The known N. S. Chernikov's Theorem [**3**], in particular, establishes: *A primitive graded group is Artinian iff it is Chernikov* (see in [**3**] Theorem B). In connection with this theorem, remind that the class of Artinian groups disagrees with the class of Chernikov groups (A. Yu. Ol'shanskiy, see, for instance, [**4**]). Further, remind that a group, in which every finitely generated (respectively 2-generator) subgroup $\neq 1$ possesses a subgroup of finite index $\neq 1$, is called locally (respectively binary) graded (S. N. Chernikov, see [**5, 6**] and [**7**, P.20] respectively). Remind also that the group G is called Shunkov, if for any its finite subgroup K, every subgroup of the factor group $N_G(K)/K$, generated by two conjugate elements of prime order, is finite (V. D. Mazurov, 1998). The class of primitive graded groups clearly includes the (extremely) large classes of binary, locally graded, Shunkov groups and (by the way) the classes of binary finite, binary solvable, linear, 2-groups, groups, having a series with binary finite factors, ...

The main results of the present article are the following theorems.

Theorem 1. *For the group G the following statements are equivalent:*

(i) G is primitive graded and satisfies $min - \bar{n}$.

(ii) G is Chernikov or Dedekind.

Theorem 2. *Let $\mathfrak{C}$ be the minimal local class of groups closed with respect to forming subgroups, series (and, at the same time, subcartesian products) and containing the class $\mathfrak{G}$ of all primitive graded groups; $G \in \mathfrak{C}$. The group G satisfies $min - \bar{n}$ iff it is Chernikov or Dedekind.*

(The class $\mathfrak{X}$ of groups is called local, if it includes every group with a local system of subgroups belonging to $\mathfrak{X}$.)

For instance, the following propositions are immediate consequences of Theorem 1.

Corollary 1. *A group, having a series with binary finite factors, satisfies $min - \bar{n}$ iff it is Chernikov or Dedekind.*

Corollary 2. *A locally graded group satisfies $min - \bar{n}$ iff it is Chernikov or Dedekind.*

Corollary 3. *A binary graded group satisfies $min - \bar{n}$ iff it is Chernikov or Dedekind.*

Below, as usual, X' is the derived subgroup of the group X; $min - ab$ is the minimal condition for abelian subgroups.

Note: *Every group, which satisfies $min - \bar{n}$ and does not satisfy $min - ab$, is necessarily Dedekind* (S. N. Chernikov, see Corollary 2 [**1**]).

Preface the proof of Theorem 1 with the following proposition.

Proposition 1. *Let G be a group satisfying $min - \bar{n}$. Then G' is Artinian. Further, G' is Chernikov iff G is Chernikov or Dedekind.*

Proof. Show that G' is Artinian. We may assume that G is non-Artinian. It is easy to see: G contains some non-Artinian subgroup H such that any its non-Artinian subgroup is normal in G; H has some descending series $H = H_0 \supset H_1 \supset H_2 \supset \ldots \supset H_\gamma \supseteq 1$ such that $H_\gamma = \cap_{\alpha<\gamma} H_\alpha$ and H_γ is Artinian.

Obviously all H/H_α with $\alpha < \gamma$ are Dedekind. Then in view of R. Baer's Theorem [**2**], for $\alpha < \gamma$, $|(H/H_\alpha)'| \leq 2$. Therefore, clearly, $|(H/H_\gamma)'| \leq 2$. Put $(H/H_\gamma)' = L/H_\gamma$. Since H_γ is Artinian and $|L : H_\gamma|$ is finite, L is Artinian.

Obviously H/L is abelian and non-Artinian. So G/L does not satisfy $min - ab$ and satisfies $min - \bar{n}$. Therefore in view of Corollary 2 [**1**], it is Dedekind. Put $(G/L)' = R/L$. Then $G' \subseteq R$. Again in view of R. Baer's Theorem [**2**], $|R : L| \leq 2$. Therefore because of L is Artinian, R is Artinian too. At the same time, G' is Artinian.

If G' is Chernikov, then, obviously, G has a series with finite factors that includes G'. Therefore in view of S. N. Chernikov's Theorem [**1**] (see above), G is Chernikov or Dedekind.

If G is Chernikov, then G' is Chernikov. If G is Dedekind, then $|G'| \leq 2$.

Proposition is proven. □

Proof of Theorem 1. Suppose that (i) is fulfilled but (ii) is not. Then by Proposition 1, G' is Artinian non-Chernikov. But in view of N. S. Chernikov's Theorem [**3**], the Artinian primitive graded subgroup G' is Chernikov, which is a contradiction.

Clearly, (ii) implies (i).

Theorem is proven. □

Corollary 4. *Let $\mathfrak{H}$ be the maximal class of groups in which every group with $min - \bar{n}$ is Chernikov or Dedekind. Then $\mathfrak{H}$ is local and closed with respect to forming series and subcartesian products.*

Proof. Indeed, it is not difficult to see: if some group G with $min - \bar{n}$ is a locally or residually $\mathfrak{H}$-group or has a series with $\mathfrak{H}$-factors, then G is locally graded. So by Corollary 2, G is Chernikov or Dedekind. □

Proof of Theorem 2. Let $\mathfrak{H}$ be from Corollary 4. Let $\mathfrak{M}$ be the minimal local class of groups closed with respect to forming series and containing $\mathfrak{G}$. In view of Corollary 4, $\mathfrak{H}$ is local and closed with respect to forming series. Further, in view of Theorem 1, $\mathfrak{G} \subseteq \mathfrak{H}$. Therefore clearly $\mathfrak{M} \subseteq \mathfrak{H}$.

Taking into account that $\mathfrak{G}$ is closed with respect to forming subgroups, it is not difficult to show that $\mathfrak{M}$ is closed with respect to forming subgroups. Indeed, obviously $\mathfrak{M}$ is the union of the classes $\mathfrak{M}_\alpha$ where $\mathfrak{M}_0 = \mathfrak{G}$ and for ordinals $\alpha > 0$ by induction: if $\alpha = \beta + 1$ for some ordinal β, then $\mathfrak{M}_\alpha$ is the class of all groups with a local system of subgroups having a series with $\mathfrak{M}_\beta$-factors; if there is no such β, then $\mathfrak{M}_\alpha = \cup_{\beta<\alpha}\mathfrak{M}_\beta$. Let us assume that all $\mathfrak{M}_\beta$, $\beta < \alpha$, are closed with respect to forming subgroups. Then it is easy to see: $\mathfrak{M}_\alpha$ is also closed with respect to forming subgroups.

Therefore clearly $\mathfrak{M} = \mathfrak{C}$. Thus $\mathfrak{C} \subseteq \mathfrak{H}$.

Let G satisfy $min - \bar{n}$. Since also $G \in \mathfrak{C}$ and $\mathfrak{C} \subseteq \mathfrak{H}$, G is Chernikov or Dedekind.

If G is Chernikov or Dedekind, then, it, of course, satisfies $min - \bar{n}$.

Theorem is proven. □

Corollary 5. *Let $\mathfrak{H}$ be from Corollary 4, and G be a group such that for every g, $h \in G$, the subgroup $< g, g^h >$ has a homomorphic image $K \neq 1$ and $K \in \mathfrak{H}$ whenever g is a p-element $\neq 1$ with some odd p and also $[g^p, h] = 1$ and $< g, h >$ is periodic. Then $G \in \mathfrak{H}$.*

Proof. Let G satisfy $min - \bar{n}$. Then K also satisfies $min - \bar{n}$. So K is Chernikov or Dedekind. At the same time, the 2-generator K has a finite homomorphic image $\neq 1$. So $< g, g^h >$ possesses a subgroup of finite index $\neq 1$. Thus G is primitive graded. In view of Theorem 1, G is Chernikov or Dedekind.

Corollary is proven. □

References

[1] *Chernikov S. N.* Infinite non-abelian groups with the minimal condition for non-normal subgroups // Mat. Zametki. — 1969. — 6, No 1. — P. 11–18 (in Russian).

[2] *Baer R.* Situation der Untergruppen und Struktur der Gruppe // S.-B. Heidelberg. Akad. — 1933. — 2. — S. 12–17.

[3] *Chernikov N. S.* Groups with minimal conditions // Fundament. i prykl. matematika. — 2008. — 14, No 5. — P. 219–235 (in Russian). Engl. transl.: J. Math. Sci. — 2009. — 163, No 6. — P. 774–784.

[4] Ol'shanskiy A. Yu. *Geometry of defining relations in groups.* — Moscow: Nauka, 1989. — 448 p. (in Russian).

[5] *Chernikov S. N.* Infinite non-abelian groups with the condition of invariance for infinite non-abelian subgroups // Dokl. Akad. Nauk SSSR. — 1970. — 194, No 6. — P. 1280–1283 (in Russian).

[6] Chernikov S. N. *Groups with prescribed properties of the system of subgroups.* — Moscow: Nauka, 1980. — 384 p. (in Russian).

[7] *Chernikov S. N.* Infinite groups defined by properties of the system of infinite subgroups // Sixth All-Union Symposium on Group Theory (Cherkassy, 1978). — Kiev: Naukova Dumka, 1980. — P. 5–22 (in Russian).