Науковий часопис НПУ імені М.П.Драгоманова. Серія 1. Фізико-математичні науки
Київ: НПУ імені М.П. Драгоманова. — 2012, № 13(1).— С. 86–108.

Characterization theorems for customer equivalent utility insurance premium calculation principle

M. V. Pratsiovytyi

National Pedagogical Dragomanov University

V. O. Drozdenko

National Pedagogical Dragomanov University

Abstract. Characterization theorems for several properties possessed by customer equivalent utility insurance premium principle are presented. Demonstrated theorems cover cases of additivity, consistency, iterativity, and scale invariance properties. We show also that for customer zero utility principle subjected to pricing of only strictly positive risks, class of utility functions producing scale invariant premiums is larger than in the general case.

Характеризаційні теореми для підрахунку вартості страхових контрактів за принципом еквівалентної корисності клієнта

М. В. Працьовитий,

Національний педагогічний університет імені М.П.Драгоманова

В. О. Дрозденко,

Національний педагогічний університет імені М.П.Драгоманова

Анотація. В роботі представлено характеризаційні теореми, що стосуються виконання декількох бажаних властивостей принципом еквівалентної корисності клієнта підрахунку вартості страхових контрактів. Представлені теореми охоплюють властивість адитивності, консистентності, ітеративності та мультиплікативної інваріантності. Ми показуємо також, що у випадку звуження принципу нульової корисності клієнта до оцінювання лише строго позитивних ризиків, клас функцій корисності, які породжують мультиплікативно-інваріантні премії є ширшим ніж у загальному випадку.

1. Introduction

Let us consider random variable X representing size of insurance compensation related to a particular insurance pact. Premium to be paid for risk X will be denoted as $\pi[X]$.

E-mail: prats4@yandex.ru, drozdenko@yandex.ru

In majority of cases random variable X is assumed to be a non-negative random variable i.e. it takes vale zero if the contract will not produce a claim and will be equal to the claim size if there will be a claim. In some case, however, negative values of variable X are also aloud; such negative values are often interpreted as compensations which have to be paid by the customer to the insurance company, for example, as penalties for interruption of contract conditions.

Let us now define several insurance premium calculation principles which we would like to investigate.

Customer equivalent utility premium for risk X, which we denote as $\pi_{\text{c.e.u.}}[X]$, is defined as solution to the equation

$$u(\omega - \pi_{\text{c.e.u.}}[X]) = \mathsf{E}[u(\omega - X)], \tag{1}$$

where ω is customer's capital at the moment when the contract is initiated, and function $u(x)$ (real valued function of real parameter) is customer's utility function i.e. it satisfies conditions $u'(x) > 0$ and $u''(x) \leq 0$ for $x \in \mathbb{R}$.

In some cases customer's utility function is selected in such a way that value $u(0)$ represents customer's utility at the moment when the contract is initiated. In such cases equation (1) for risk X is replaced by the equation

$$u(-\pi[X]) = \mathsf{E}[u(-X)] \tag{2}$$

and corresponding method of pricing of insurance contracts is called *customer zero utility premium calculation principle*. Obtained in such a way premium in the article will be denoted as $\pi_{\text{c.z.u.}}[X]$.

Sometimes customer equivalent utility premium calculation principle and customer zero utility premium calculation principle are applied to some special classes of risks: as an example of such a class one can mention class of all non-negative risks, alternatively one could mention class of non-negative risks bounded from above by some fixed real value, etc. In such cases domain of function $u(x)$ could be a subset of $\mathbb{R}$ such that equation (1) or, alternatively, equation (2), depending on chosen method of pricing, will preserve its correct mathematical meaning, moreover, monotonicity and concavity properties of function $u(x)$ should also be preserved.

Net premium for risk X, which in the article will be denoted as $\pi_{\text{net}}[X]$, is defined as expected value of losses associated with risk X i.e.

$$\pi_{\text{net}}[X] = \mathsf{E}[X].$$

Exponential premium, dependent on parameter β, for risk X which in the article will be denoted as $\pi_{\exp(\beta)}[X]$, is defined in the following way

$$\pi_{\exp(\beta)}[X] = \frac{1}{\beta}\log\left(\mathsf{E}[e^{\beta X}]\right), \quad \text{for} \quad \beta > 0.$$

We will say that a premium calculation principle $\pi[X]$ possesses:
additivity property if for any two independent risks X_1 and X_2 holds identity

$$\pi[X_1 + X_2] = \pi[X_1] + \pi[X_2]; \tag{3}$$

consistency property if for any risk X and any real constant c (if a pricing method is defined only for non-negative risks then constant c can be claimed to be non-negative in order to avoid situations when $X + c < 0$ i.e. situations when value $\pi[X + c]$ is undefined) holds identity

$$\pi[X + c] = \pi[X] + c; \tag{4}$$

iterativity property if for any two risks X and Y holds identity

$$\pi[\pi[X|Y]] = \pi[X]; \tag{5}$$

scale invariance property if for any risk X and any positive real constant Θ holds identity

$$\pi[\Theta X] = \Theta\,\pi[X]; \tag{6}$$

property of no unjustified risk loading if for any risk X such that $\mathsf{P}\{X = C\} = 1$ for some real constant C (different constants can be chosen for different risks), the following equation holds

$$\pi[X] = C. \tag{7}$$

More information about defined methods of pricing of insurance contracts as well as properties that can be possessed by insurance premium calculation principles can be found, for example, in Asmussen and Albrecher (2010), Boland (2007), Bowers et al (1997), Bühlmann (1970), Dickson (2005), Gerber (1979), De Vylder et al (1984), De Vylder et al (1986), Kaas et al (2008), Kremer (1999), Rolski et al (1998), Straub (1988).

We would like to mention that research related to theorems of characterization type for properties possessed by certain insurance premium calculation principles was initiated by the Swiss mathematician Hans-Ulrich Gerber, see Gerber (1979). Gerber himself has proven characterization theorems for consistency and additivity properties of mean value premium principle and characterization theorem for iterativity property of insurer zero utility premium principle. Corresponding theorems for customer equivalent utility premium calculation principle and customer zero utility premium principle were still missing in the literature.

2. Additivity Property

The following theorem describes necessary and sufficient conditions for attainment of additivity property by customer equivalent utility premium calculation principle.

1. *Customer equivalent utility premium calculation principle possesses additivity property if and only if* $u(x) = ax + b$, *for* $a > 0$, *or* $u(x) = -\alpha e^{-\beta x} + \gamma$, *for* $\min[\alpha, \beta] > 0$, *i.e. only in the cases when it coincides either with net premium principle or with exponential premium principle.*

Observe that class of functions $u(x) = -\alpha e^{-\beta x} + \gamma$, for $\min[\alpha, \beta] > 0$, is quite rich. It contains, for example, all functions of the form $u(x) = -\tau^{-x}$, for some real constant $\tau > 1$, because function $u(x) = -\tau^{-x}$ can be represented as $u(x) = -e^{-\log(\tau)\,x}$, this means that in the considered case $\beta := \log(\tau)$.

Proof. Let us from the beginning prove sufficiency of the statement. We start from the case of $u(x) = ax + b$, for $a > 0$. Indeed, in this case for any two independent risks X_1 and X_2, and any customer's initial capital ω, from equation (1) follows

$$a(\omega - \pi_{\text{c.e.u.}}[X_i]) + b = \mathsf{E}[a(\omega - X_i) + b], \quad \text{for} \quad i = \overline{1,\,2},$$

thus

$$\pi_{\text{c.e.u.}}[X_i] = \mathsf{E}[X_i] = \pi_{\text{net}}[X_i], \quad \text{for} \quad i = \overline{1,\,2}.$$

On the other hand, from the same equation follows

$$a(\omega - \pi_{\text{c.e.u.}}[X_1 + X_2]) + b = \mathsf{E}[a(\omega - X_1 - X_2) + b],$$

hence

$$\pi_{\text{c.e.u.}}[X_1 + X_2] = \mathsf{E}[X_1] + \mathsf{E}[X_2] = \pi_{\text{c.e.u.}}[X_1] + \pi_{\text{c.e.u.}}[X_2],$$

so, we could see that customer equivalent utility premium calculation principle possesses additivity property in the case of linear customer's utility function.

Let us now switch to the case of $u(x) = -\alpha e^{-\beta x} + \gamma$, for $\min[\alpha, \beta] > 0$. Here for any independent risks X_1 and X_2, and any customer's initial capital ω, we get

$$-\alpha e^{-\beta(\omega - \pi_{\text{c.e.u.}}[X_i])} + \gamma = \mathsf{E}[-\alpha e^{-\beta(\omega - X_i)} + \gamma], \quad \text{for} \quad i = \overline{1,\,2},$$

which yields

$$\pi_{\text{c.e.u.}}[X_i] = \frac{1}{\beta}\log(\mathsf{E}[e^{\beta X_i}]) = \pi_{\exp(\beta)}[X_i], \quad \text{for} \quad i = \overline{1,\,2}.$$

Moreover

$$-\alpha e^{-\beta(\omega - \pi_{\text{c.e.u.}}[X_1 + X_2])} + \gamma = \mathsf{E}[-\alpha e^{-\beta(\omega - X_1 - X_2)} + \gamma] = -\alpha e^{-\beta\omega}\mathsf{E}[e^{\beta X_1}]\mathsf{E}[e^{\beta X_2}] + \gamma,$$

hence

$$\pi_{\text{c.e.u.}}[X_1+X_2]=\frac{1}{\beta}\log(\mathsf{E}[e^{\beta X_1}])+\frac{1}{\beta}\log(\mathsf{E}[e^{\beta X_2}])=\pi_{\text{c.e.u.}}[X_1]+\pi_{\text{c.e.u.}}[X_2],$$

and as we have seen, additivity property is possessed by customer equivalent utility premium calculation principle in the case of exponential customer's utility function.

Proof of the sufficiency was finished, so we can start to prove the necessity.

Observe that customer equivalent utility premium calculation principle is invariant with respect to linear transformations of function $u(x)$, i.e. principle based on utility function $u(x)$ and principle based on utility function $\bar{u}(x)=l_1u(x)+l_2$, for $l_1>0$, will produce the same premiums. Here condition $l_1>0$ is imposed because otherwise the assumption of positivity of first derivative of function $\bar{u}(x)$ will vanish.

In order to simplify the computations, we will fix value of customer's initial capital ω, derive all possible representations (in the case when customer equivalent utility principle is additive) for function $\bar{u}(x)$ with

$$l_1=1/u'(\omega)\quad\text{and}\quad l_2=-u(\omega)/u'(\omega),$$

and then we will switch back to function $u(x)$.

Observe that just defined utility function $\bar{u}(x)$ satisfies the following boundary conditions

$$\bar{u}(\omega)=0,\quad \bar{u}'(\omega)=1,\quad\text{and}\quad \bar{u}''(\omega)=\kappa, \tag{8}$$

for some constant $\kappa\le 0$.

Let us now consider risk X which takes only two possible values, namely t (here t is any real number different from zero) and 0 with probabilities p and $1-p$ respectively. Risk X can be viewed as a random function of parameters p and t, and, therefore, within the proof of Theorem 1 will be denoted as X_p^t.

Equation (1) based on utility function $\bar{u}(x)$ for risk X_p^t will have a form

$$\bar{u}(\omega-\pi_{\text{c.e.u.}}[X_p^t])=\bar{u}(\omega-t)p+\bar{u}(\omega)(1-p). \tag{9}$$

Putting $p=0$ into (9), obtain

$$\bar{u}(\omega-\pi_{\text{c.e.u.}}[X_0^t])=\bar{u}(\omega). \tag{10}$$

Since $\bar{u}'(x)>0$ for all x, then from (10) follows

$$\pi_{\text{c.e.u.}}[X_0^t]=0. \tag{11}$$

Let us calculate partial derivatives with respect to p from both sides of equation (9)

$$-\bar{u}'(\omega-\pi_{\text{c.e.u.}}[X_p^t])\cdot\frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t]=\bar{u}(\omega-t)-\bar{u}(\omega). \tag{12}$$

Putting $p = 0$ into equation (12), get

$$-\bar{u}'(\omega - \pi_{\text{c.e.u.}}[X_0^t]) \cdot \frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t]\bigg|_{p=0} = \bar{u}(\omega - t) - \bar{u}(\omega). \tag{13}$$

Combination of (11) and (13) implies

$$-\bar{u}'(\omega) \cdot \frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t]\bigg|_{p=0} = \bar{u}(\omega - t) - \bar{u}(\omega). \tag{14}$$

Substituting boundary conditions $\bar{u}(\omega) = 0$ and $\bar{u}'(\omega) = 1$ into equation (14), we obtain representation for partial derivative with respect to parameter p of the premium at point $p = 0$, namely,

$$\frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t]\bigg|_{p=0} = -\bar{u}(\omega - t). \tag{15}$$

Let us consider also risk Y, independent of X, taking two possible values, namely h (here h is any non-zero real number) and 0 with probabilities q and $1 - q$ respectively. Being a random function of parameters h and q risk Y will be denoted as Y_q^h.

Using manipulations similar to those performed with risk X_p^t, one can conclude that

$$\pi_{\text{c.e.u.}}[Y_0^h] = 0, \tag{16}$$

and that partial derivative with respect to parameter q of the premium at point $q = 0$ is

$$\frac{\partial}{\partial q}\pi_{\text{c.e.u.}}[Y_q^h]\bigg|_{q=0} = -\bar{u}(\omega - h). \tag{17}$$

Now let us look at risk $Z_{p,q}^{t,h}$ defined in the following way

$$Z_{p,q}^{t,h} := X_p^t + Y_q^h.$$

Risk $Z_{p,q}^{t,h}$ will take values $t + h$, t, h, and 0 with probabilities $p\,q$, $p\,(1 - q)$, $(1 - p)q$, and $(1 - p)(1 - q)$ respectively.

If customer equivalent utility principle is additive then the following must hold

$$\pi_{\text{c.e.u.}}[Z_{p,q}^{t,h}] = \pi_{\text{c.e.u.}}[X_p^t] + \pi_{\text{c.e.u.}}[Y_q^h].$$

In this case equation (1) for risk $Z_{p,q}^{t,h}$ based on function $\bar{u}(x)$ will have a form

$$\begin{aligned}\bar{u}(\omega - \pi_{\text{c.e.u.}}[X_p^t] - \pi_{\text{c.e.u.}}[Y_q^h]) \;&=\; \bar{u}(\omega - t - h)p\,q + \\ &+ \bar{u}(\omega - t)p\,(1 - q) + \bar{u}(\omega - h)(1 - p)q + \bar{u}(\omega)(1 - p)(1 - q).\end{aligned} \tag{18}$$

Using boundary condition $\bar{u}(\omega) = 0$ equation (18) can be slightly simplified to the following

$$\begin{aligned}\bar{u}(\omega - \pi_{\text{c.e.u.}}[X_p^t] - \pi_{\text{c.e.u.}}[Y_q^h]) \;&=\; \bar{u}(\omega - t - h)p\,q + \\ &+\; \bar{u}(\omega - t)p\,(1 - q) + \bar{u}(\omega - h)(1 - p)q.\end{aligned} \tag{19}$$

Let us calculate partial derivatives with respect to p from both sides of equation (19)

$$\begin{aligned}- \bar{u}'(\omega - \pi_{\text{c.e.u.}}[X_p^t] - \pi_{\text{c.e.u.}}[Y_q^h]) \cdot \frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t] \ &= \\ = \ \bar{u}(\omega - t - h)q + \bar{u}(\omega - t)(1-q) - \bar{u}(\omega - h)q.&\end{aligned} \tag{20}$$

Next step is to take partial derivatives with respect to q from both sides of equation (20), obtain

$$\begin{aligned}\bar{u}''(\omega - \pi_{\text{c.e.u.}}[X_p^t] - \pi_{\text{c.e.u.}}[Y_q^h]) \cdot \frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t] \cdot \frac{\partial}{\partial q}\pi_{\text{c.e.u.}}[Y_q^h] \ &= \\ = \ \bar{u}(\omega - t - h) - \bar{u}(\omega - t) - \bar{u}(\omega - h).&\end{aligned} \tag{21}$$

Substituting $p = q = 0$ into equation (21) and using identities (11), (15), (16), and (17), as well as boundary condition $\bar{u}''(\omega) = \kappa$, we finally get equation which function $\bar{u}(\cdot)$ has to satisfy for customer equivalent utility principle to be additive

$$\kappa\, \bar{u}(\omega - t)\, \bar{u}(\omega - h) = \bar{u}(\omega - t - h) - \bar{u}(\omega - t) - \bar{u}(\omega - h). \tag{22}$$

Solving equation (22), we will investigate separately cases of $\kappa = 0$ and $\kappa < 0$.

In the case of $\kappa = 0$ equation (22) will be simplified to the following

$$\bar{u}(\omega - t - h) = \bar{u}(\omega - t) + \bar{u}(\omega - h). \tag{23}$$

Taking partial derivatives with respect to h from both sides of equation (23), obtain

$$\bar{u}'(\omega - t - h) = \bar{u}'(\omega - h). \tag{24}$$

Since function $\bar{u}(\cdot)$ was assumed to be twice differentiable, then function $\bar{u}'(\cdot)$ must be continuous. Therefore must exist limit of $\bar{u}'(\omega - h)$ as h tends to zero and it has to be equal to $\bar{u}'(\omega)$. Taking limits as h tends to zero from both sides of equation (24) and using boundary condition $\bar{u}'(\omega) = 1$, obtain

$$\bar{u}'(\omega - t) = 1. \tag{25}$$

Parameter t was taken from $\mathbb{R} \setminus \{0\}$, however due to continuity (which follows from differentiability, since function $\bar{u}(\cdot)$ is twice differentiable) of function $\bar{u}'(\cdot)$, equation (25) can be rewritten in terms of parameter $x \in \mathbb{R}$.

Using (25) and boundary condition $\bar{u}(\omega) = 0$ we finally get first admissible representation for function $\bar{u}(x)$, namely,

$$\bar{u}(x) = x - \omega. \tag{26}$$

Combining (26) with transformation identity

$$\bar{u}(x) = l_1 u(x) + l_2, \quad \text{for} \quad l_1 = 1/u'(\omega) \quad \text{and} \quad l_2 = -u(\omega)/u'(\omega), \tag{27}$$

we finally get corresponding admissible representation for original utility function $u(x)$

$$u(x) = u'(\omega)(x - \omega) + u(\omega). \tag{28}$$

Equation (28) means that tangent straight line to function $u(x)$ at point ω coincides with function $u(x)$ itself, hence function $u(x)$ must be a function of the form

$$u(x) = ax + b$$

for some constants a and b. Moreover, constant a must be a strictly positive constant because otherwise this would contradict with the assumption of positivity of first derivative of function $u(x)$.

Let us now consider case of $\kappa < 0$. Taking repeatedly partial derivatives with respect to t and then with respect to h from both sides of (22), we get equation

$$\kappa\, \bar{u}'(\omega - t)\, \bar{u}'(\omega - h) = \bar{u}''(\omega - t - h) \tag{29}$$

Due to continuity of $\bar{u}'(\cdot)$, one can define $\bar{u}'(\omega)$ as $\lim_{h\to 0} \bar{u}'(\omega - h)$, then, switching to the limit as h tends to zero from both sides of equation (29), and using boundary condition $\bar{u}'(\omega) = 1$, obtain

$$\bar{u}''(\omega - t) = \kappa\, \bar{u}'(\omega - t). \tag{30}$$

From equation (31) and boundary condition $\bar{u}'(\omega) = 1$, follows

$$\bar{u}'(\omega - t) = e^{\kappa\,(\omega - t)} \cdot e^{-\kappa\,\omega}. \tag{31}$$

Using (31) and boundary condition $\bar{u}(\omega) = 0$ we finally get admissible representation for utility function $\bar{u}(\cdot)$ in the case of $\kappa < 0$, namely,

$$\bar{u}(\omega - t) = \frac{e^{\kappa\,(\omega - t)} \cdot e^{-\kappa\,\omega} - 1}{\kappa},$$

or equivalently, again due to continuity of $\bar{u}(\cdot)$, in terms of parameter $x \in \mathbb{R}$,

$$\bar{u}(x) = \frac{e^{\kappa\,x} \cdot e^{-\kappa\,\omega} - 1}{\kappa}. \tag{32}$$

Taking into account that $\bar{u}''(\omega) = \kappa$, using representation (32) and transformation identity (27), we finally get corresponding admissible representation for original utility function $u(x)$, or more precisely,

$$u(x) = \frac{u'(\omega)\, e^{-\bar{u}''(\omega)\cdot\omega}}{\bar{u}''(\omega)} \cdot e^{\bar{u}''(\omega)\cdot x} - \frac{u'(\omega)}{\bar{u}''(\omega)} + u(\omega). \tag{33}$$

From representation (33) follows that in the case of $\bar{u}''(\omega) < 0$ function $u(x)$ must be a function of the form

$$u(x) = -\alpha e^{-\beta x} + \gamma$$

for some constants α, β, and γ. Moreover conditions $u'(\omega) > 0$ and $\bar{u}''(\omega) < 0$ imply additional restrictions on parameters α and β, namely, both of them must be strictly positive constants, or equivalently, $\min[\alpha, \beta] > 0$.

This completes the proof of Theorem 1. $\square$

Let us now show that customer equivalent utility premium principle coincides with net premium principle if and only if $u(x) = ax + b$, for $a > 0$, and coincides with exponential premium principle if and only if $u(x) = -\alpha e^{-\beta x} + \gamma$, for $\min[\alpha,\ \beta] > 0$.

For this reason we will need inequality

$$\pi_{\exp(\beta)}[X] \ = \ \frac{1}{\beta}\log(\mathsf{E}[e^{\beta X}]) \ \geq \ \frac{1}{\beta}\log(e^{\beta\,\mathsf{E}[X]}) \ = \ \mathsf{E}[X] \ = \ \pi_{\mathrm{net}}[X],$$

and, moreover, exact equality in the inequality $\mathsf{E}[e^{\beta X}] \geq e^{\beta\,\mathsf{E}[X]}$ appears if and only if $\mathsf{P}\{X = C\} = 1$ for some constant $C \in \mathbb{R}$. Therefore, generally speaking, net premium principle is not a special case of exponential premium principle and viceversa.

Let us now assume that for some function $u(x)$, different from exponential function, customer equivalent utility premium principle will be equivalent to exponential premium principle. Then, due to additivity of exponential premium principle, such method of pricing must be additive. However in the proof of Theorem 1 was shown that customer equivalent utility premium calculation principle is additive if and only if $u(x) = ax + b$, for $a > 0$, and $u(x) = -\alpha e^{-\beta x} + \gamma$, for $\min[\alpha,\ \beta] > 0$. Here case of $u(x) = ax + b$, for $a > 0$, correspond to net premium principle, which, as was demonstrated, generally speaking is not a special case of exponential premium principle. As we see, original assumption about existence of non-exponential customer's utility function $u(x)$ which would produce a principle equivalent to the exponential premium principle leads to a contradiction. Therefore, case of $u(x) = -\alpha e^{-\beta x} + \gamma$, for $\min[\alpha,\ \beta] > 0$, is indeed the only case when customer equivalent utility premium principle is equivalent to exponential premium principle.

Using similar contradiction technique one can conclude that the case of $u(x) = ax + b$, for $a > 0$, is the only case when customer equivalent utility premium principle is equivalent to net premium principle.

Since customer's initial capital in the proof of Theorem 1 was chosen arbitrary and no restriction on it had been used within the proof, then we can formulate the following corollary to Theorem 1.

1. *Customer zero utility premium calculation principle possesses additivity property if and only if* $u(x) = ax + b$, *for* $a > 0$, *or* $u(x) = -\alpha e^{-\beta x} + \gamma$, *for* $\min[\alpha, \beta] > 0$, *i.e. only in the cases when it coincides either with net premium principle or with exponential premium principle.*

Alternatively one can prove Theorem 1 in the following way. Sufficiency, as usual, can be shown by direct checking. Then, in order to show the necessity, we show first that customer equivalent utility premium calculation principle possesses property of no unjustified risk loading. Indeed, for any risk X such that $\mathsf{P}\{X = C\} = 1$ (here C is some real constant), for any customer's initial capital ω and any customer's utility function $u(x)$, equivalent utility equation (1) will have a form

$$u(\omega - \pi_{\text{c.e.u.}}[X]) \;=\; \mathsf{E}[u(\omega - X)] \;=\; u(\omega - C). \tag{34}$$

Since customer's utility function $u(x)$ is a strictly increasing function, then from equation (34) follows

$$\pi_{\text{c.e.u.}}[X] = C,$$

hence property of no unjustified risk loading is possessed. Then, before proving Theorem 1 one have to prove Theorem 2. If customer equivalent utility principle for some utility function $u(x)$ is additive, then by definition identity (3) must hold for any two independent risks X_1 and X_2. Therefore, for any risk X and any real constant C in the case of additive customer equivalent utility premium calculation principle must hold identity

$$\pi_{\text{c.e.u.}}[X + C] = \pi_{\text{c.e.u.}}[X] + \pi_{\text{c.e.u.}}[C]. \tag{35}$$

Due to property of no unjustified risk loading, equation (35) can be rewritten in the following form

$$\pi_{\text{c.e.u.}}[X + C] = \pi_{\text{c.e.u.}}[X] + C.$$

This means that additive customer equivalent utility premium calculation principle has to be consistent. Hence from Theorem 2 follows that customer's utility function $u(x)$ can only be a function of the forms $u(x) = ax + b$, for $a > 0$, or $u(x) = -\alpha e^{-\beta x} + \gamma$, for $\min[\alpha, \beta] > 0$.

3. Consistency Property

The following theorem describes necessary and sufficient conditions imposed on customer's utility function under which consistency property is possessed by customer equivalent utility premium calculation principle.

2. *Customer equivalent utility premium calculation principle possesses consistency property if and only if $u(x) = ax + b$, for $a > 0$, or $u(x) = -\alpha e^{-\beta x} + \gamma$, for $\min[\alpha, \beta] > 0$, i.e. only in the cases when it coincides either with net premium principle or with exponential premium principle.*

PROOF. Let us from the beginning prove sufficiency of the statement. We start from linear utility function $u(x) = ax + b$, for $a > 0$, and show that in this case customer equivalent utility premium principle is equivalent to net principle. Indeed, here for any risk X, and any customer's initial capital ω, equation (1) will be simplified to the following

$$a(\omega - \pi_{\text{c.e.u.}}[X]) + b = \mathsf{E}[a(\omega - X) + b],$$

therefore, in the case of linear customer's utility function,

$$\pi_{\text{c.e.u.}}[X] = \mathsf{E}[X] = \pi_{\text{net}}[X].$$

Moreover, for any real constant c, the same risk, the same customer's initial capital, and the same linear customer's utility function, form equation (1) we get

$$a(\omega - \pi_{\text{c.e.u.}}[X + c]) + b = \mathsf{E}[a(\omega - X - c) + b],$$

hence, in the considered case holds identity

$$\pi_{\text{c.e.u.}}[X + c] = \mathsf{E}[X] + c = \pi_{\text{c.e.u.}}[X] + c,$$

and as we see, customer equivalent utility premium calculation principle possesses consistency property in the case of linear customer's utility function.

Let us now switch to the case of $u(x) = -\alpha e^{-\beta x} + \gamma$, for $\min[\alpha, \beta] > 0$. We show first that it is the case when customer equivalent utility premium principle is equivalent to exponential premium principle. Indeed, here for any risk X from equation (1) we get

$$-\alpha e^{-\beta(\omega - \pi_{\text{c.e.u.}}[X])} + \gamma = \mathsf{E}[-\alpha e^{-\beta(\omega - X)} + \gamma],$$

which yields

$$\pi_{\text{c.e.u.}}[X] = \frac{1}{\beta}\log(\mathsf{E}[e^{\beta X}]) = \pi_{\exp(\beta)}[X].$$

Then again, for any real constant c, the same risk, the same customer's initial capital, and the same exponential customer's utility function from equation (1) we get

$$-\alpha e^{-\beta(\omega - \pi_{\text{c.e.u.}}[X+c])} + \gamma = \mathsf{E}[-\alpha e^{-\beta(\omega - X - c)} + \gamma],$$

hence

$$\pi_{\text{c.e.u.}}[X + c] = \frac{1}{\beta}\log(\mathsf{E}[e^{\beta X}]) + c = \pi_{\text{c.e.u.}}[X] + c$$

so we have seen that customer equivalent utility premium calculation principle possesses consistency property in the case of exponential customer's utility function.

Proof of the sufficiency was finished, so can start to prove the necessity.

Due to the invariance of customer equivalent utility premium calculation principle with respect to linear transformations of customer's utility function, like in the proof of the previous theorem, we obtain first admissible representations for utility function $\bar{u}(x)$ defined as

$$\bar{u}(x) = l_1 u(x) + l_2 \quad \text{with} \quad l_1 = 1/u'(\omega) \quad \text{and} \quad l_2 = -u(\omega)/u'(\omega), \tag{36}$$

and then will switch back to original utility function $u(x)$.

Let us remind that defined in such a way utility function $\bar{u}(x)$ will satisfy the following boundary conditions

$$\bar{u}(\omega) = 0, \quad \bar{u}'(\omega) = 1, \quad \text{and} \quad \bar{u}''(\omega) = \kappa,$$

for some constant $\kappa \leq 0$.

Let us now consider risk X taking only two possible values, namely t (here t is any real number different from zero) and 0 with probabilities p and $1-p$ respectively. Being a random function of parameters t and p, risk X within the proof of Theorem 2 will be denoted as X_p^t.

Customer equivalent utility equation (1) based on utility function $\bar{u}(x)$ for risk X_p^t will have a form

$$\bar{u}(\omega - \pi_{\text{c.e.u.}}[X_p^t]) = \bar{u}(\omega - t) \cdot p + \bar{u}(\omega) \cdot (1-p). \tag{37}$$

Substituting $p = 0$ into equation (37), obtain

$$\bar{u}(\omega - \pi_{\text{c.e.u.}}[X_0^t]) = \bar{u}(\omega). \tag{38}$$

Since $\bar{u}(\cdot)$ is a strictly increasing function, then from equation (38) follows

$$\pi_{\text{c.e.u.}}[X_0^t] = 0. \tag{39}$$

Let us now calculate partial derivatives with respect to parameter p from both sides of equation (37), obtain

$$-\bar{u}'(\omega - \pi_{\text{c.e.u.}}[X_p^t]) \cdot \frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t] = \bar{u}(\omega - t) - \bar{u}(\omega). \tag{40}$$

Substituting $p = 0$ into equation (40), using identity (39), as well as boundary conditions $\bar{u}(\omega) = 0$ and $\bar{u}'(\omega) = 1$, we get representation for partial derivative of the premium with respect to parameter p at point $p = 0$, namely

$$\left.\frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t]\right|_{p=0} = -\bar{u}(\omega - t). \tag{41}$$

Next step is to take partial derivative with respect to p from both sides of equation (40), here we get

$$\bar{u}''(\omega - \pi_{\text{c.e.u.}}[X_p^t]) \cdot \left(\frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t]\right)^2 - \bar{u}'(\omega - \pi_{\text{c.e.u.}}[X_p^t]) \cdot \frac{\partial^2}{(\partial p)^2}\pi_{\text{c.e.u.}}[X_p^t] = 0. \tag{42}$$

Substituting value $p = 0$ into equation (42) and then using identities (39) and (41) as well as boundary conditions $\bar{u}'(\omega) = 1$ and $\bar{u}''(\omega) = \kappa$ we get representation for the second partial derivative of the premium with respect to parameter p at point $p = 0$, namely

$$\left.\frac{\partial^2}{(\partial p)^2}\pi_{\text{c.e.u.}}[X_p^t]\right|_{p=0} = \kappa\, \bar{u}^2(\omega - t). \tag{43}$$

In the case of consistent customer equivalent utility premium calculation principle must hold identity

$$\pi_{\text{c.e.u.}}[X_p^t + c] = \pi_{\text{c.e.u.}}[X_p^t] + c, \quad \text{for } c \in \mathbb{R},$$

therefore, equivalent utility equation (1) based on utility function $\bar{u}(x)$ for risk $X_p^t + c$ will have a form

$$\bar{u}(\omega - \pi_{\text{c.e.u.}}[X_p^t] - c) = \bar{u}(\omega - t - c) \cdot p + \bar{u}(\omega - c) \cdot (1 - p). \tag{44}$$

Let us now calculate partial derivatives with respect to parameter p from both sides of equation (44)

$$-\bar{u}'(\omega - \pi_{\text{c.e.u.}}[X_p^t] - c) \cdot \frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t] = \bar{u}(\omega - t - c) - \bar{u}(\omega - c). \tag{45}$$

Next step is to take partial derivatives with respect to parameter p ones more, this time from both sides of equation (45), here we get

$$\begin{aligned} \bar{u}''(\omega - \pi_{\text{c.e.u.}}[X_p^t] - c) \cdot \left(\frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t]\right)^2 - \\ - \bar{u}'(\omega - \pi_{\text{c.e.u.}}[X_p^t] - c) \cdot \frac{\partial^2}{(\partial p)^2}\pi_{\text{c.e.u.}}[X_p^t] = 0 \end{aligned} \tag{46}$$

Substituting $p = 0$ into equation (46) and using identities (39) and (41), obtain

$$\bar{u}''(\omega - c) \cdot \bar{u}^2(\omega - t) - \bar{u}'(\omega - c) \cdot \frac{\partial^2}{(\partial p)^2}\pi_{\text{c.e.u.}}[X_p^t]\bigg|_{p=0} = 0. \tag{47}$$

Since utility function $\bar{u}(x)$ is a strictly increasing function (i.e. $\bar{u}'(x)$ always takes strictly positive values) then with out of loss of generality equation (47) can be rewritten in the following equivalent form

$$\frac{\partial^2}{(\partial p)^2}\pi_{\text{c.e.u.}}[X_p^t]\bigg|_{p=0} = \frac{\bar{u}''(\omega - c) \cdot \bar{u}^2(\omega - t)}{\bar{u}'(\omega - c)}. \tag{48}$$

Observe that equations (43) and (48) have equal left hand sides, hence their right hand sides also have to be equal. In this way we get equation

$$\kappa\, \bar{u}^2(\omega - t) = \frac{\bar{u}''(\omega - c) \cdot \bar{u}^2(\omega - t)}{\bar{u}'(\omega - c)}, \tag{49}$$

which after cancelation of $\bar{u}^2(\omega - t)$ multiplier will be simplified to the following

$$\frac{\bar{u}''(\omega - c)}{\bar{u}'(\omega - c)} = \kappa, \quad \text{for all } c \in \mathbb{R}. \tag{50}$$

Equation (50) is the equation which utility function $\bar{u}(\cdot)$ has to satisfy for the customer equivalent utility premium calculation principle to be consistent. We will solve the equation separately for the cases of $\kappa = 0$ and $\kappa < 0$.

In the case of $\kappa = 0$ equation (50) will be simplified to the following

$$\bar{u}''(\omega - c) = 0.$$

This means that in the mentioned case function $\bar{u}(x)$ must be a function of the form

$$\bar{u}(x) = \bar{a}x + \bar{b}$$

for some constants $\bar{a}$ and $\bar{b}$. Using boundary conditions $\bar{u}(\omega) = 0$ and $\bar{u}'(\omega) = 1$ we find out values of constants $\bar{a}$ and $\bar{b}$, or more precisely,

$$\bar{a} = 1 \quad \text{and} \quad \bar{b} = -\omega.$$

This gives us exact admissible representation for utility function $\bar{u}(x)$, namely,

$$\bar{u}(x) = x - \omega. \tag{51}$$

Combining representation (51) with transformation relation (36) we get relation containing corresponding original utility function $u(x)$

$$x - \omega = \frac{u(x)}{u'(\omega)} - \frac{u(\omega)}{u'(\omega)},$$

or equivalently

$$u(x) = u'(\omega)x + u(\omega) - \omega\, u'(\omega),$$

therefore function $u(x)$ must be a function of the form

$$u(x) = ax + b$$

for some constants a and b. Assumption of positivity of first derivative of function $u(x)$ gives us additional restriction on parameter a: parameter a must be a strictly positive constant.

Let us now consider case of $\kappa < 0$. It seems to be convenient to rewrite equation (50) in the following form

$$\bar{u}''(\omega - c) = \kappa\, \bar{u}'(\omega - c). \tag{52}$$

From equation (52), using boundary condition $\bar{u}'(\omega) = 1$, obtain

$$\bar{u}'(\omega - c) = e^{\kappa\,(\omega - c)} \cdot e^{-\kappa\,\omega}. \tag{53}$$

Using equation (53) and boundary condition $\bar{u}(\omega) = 0$ we get second admissible representation for utility function $\bar{u}(\cdot)$, namely,

$$\bar{u}(\omega - c) = \frac{e^{\kappa\,(\omega - c)} \cdot e^{-\kappa\,\omega} - 1}{\kappa}, \quad \text{for } c \in \mathbb{R},$$

or equivalently in terms of parameter x,

$$\bar{u}(x) = \frac{e^{\kappa\,x} \cdot e^{-\kappa\,\omega} - 1}{\kappa}, \quad \text{for } x \in \mathbb{R}. \tag{54}$$

Taking into account that $\bar{u}''(\omega) = \kappa$, using representation (54) and transformation relation (36), we finally get corresponding admissible representation for the original utility function, namely, utility function $u(x)$,

$$u(x) = \frac{u'(\omega)\, e^{-\bar{u}''(\omega)\cdot\omega}}{\bar{u}''(\omega)} \cdot e^{\bar{u}''(\omega)\cdot x} - \frac{u'(\omega)}{\bar{u}''(\omega)} + u(\omega). \tag{55}$$

From representation (55) follows that in the case of $\bar{u}''(\omega) < 0$ function $u(x)$ must be a function of the form

$$u(x) = -\alpha e^{-\beta x} + \gamma$$

for some constants α, β, and γ. Moreover conditions $u'(\omega) > 0$ and $\bar{u}''(\omega) < 0$ imply additional restrictions on parameters α and β, namely, both of them must be strictly positive constants, or equivalently, $\min[\alpha, \beta] > 0$.

This completes the proof of Theorem 2. □

In a way similar to one presented in the previous section, proof of Theorem 2 can be used for showing that case of $u(x) = ax + b$, for $a > 0$, is the only case when customer equivalent utility premium principle is equivalent to net premium principle and that $u(x) = -\alpha e^{-\beta x} + \gamma$, for $\min[\alpha, \beta] > 0$, is the only case when customer equivalent utility premium principle is equivalent to exponential premium principle.

Since we did not use any restrictions on customer's initial capital within the proof of Theorem 2, then we can formulate the following corollary to Theorem 2.

2. *Customer zero utility premium calculation principle possesses consistency property if and only if* $u(x) = ax + b$, *for* $a > 0$, *or* $u(x) = -\alpha e^{-\beta x} + \gamma$, *for* $\min[\alpha, \beta] > 0$, *i.e. only in the cases when it coincides either with net premium principle or with exponential premium principle.*

4. Iterativity Property

In contrast to insurer equivalent utility premium calculation principle which possesses iterativity property only in the cases of exponential and linear utility functions, customer equivalent utility premium calculation principle possesses iterativity property with arbitrary choice of the admissible customer's utility function. We believe that this observation deserves to be formulated in a form of theorem.

3. *Customer's equivalent utility premium calculation principle possesses iterativity property for arbitrary choice of the initial capital* ω *and arbitrary choice of the utility function* $u(x) \in C_2(\mathbb{R})$ *such that* $u'(x) > 0$ *and* $u''(x) \leq 0$ *for all* $x \in \mathbb{R}$.

Proof. Here for any two risks X and Y, any customer's initial capital ω as well as any admissible customer's utility function $u(x)$, we get

$$\begin{aligned}
\pi_{\text{c.e.u.}}[\pi_{\text{c.e.u.}}[X|Y]] &= -u^{-1}(\mathsf{E}[u(\omega - \pi_{\text{c.e.u.}}[X|Y])]) + \omega \\
&= -u^{-1}(\mathsf{E}[u(\omega + u^{-1}(\mathsf{E}[u(\omega - X)|Y]) - \omega)]) + \omega \\
&= -u^{-1}(\mathsf{E}[\mathsf{E}[u(\omega - X)|Y]]) + \omega \\
&= -u^{-1}(\mathsf{E}[u(\omega - X)]) + \omega \quad = \quad \pi_{\text{c.e.u.}}[X].
\end{aligned}$$

Hence statement of Theorem 3 indeed holds. □

Since customer's initial capital in the proof of Theorem 3 was chosen arbitrary and no restrictions on it had been used within the proof, then we can formulate the following corollary to Theorem 3.

3. *Customer's zero utility premium calculation principle always possesses iterativity property for arbitrary choice of customer's utility function* $u(x) \in C_2(\mathbb{R})$ *such that* $u'(x) > 0$ *and* $u''(x) \leq 0$ *for all* $x \in \mathbb{R}$.

5. Scale Invariance Property

The following theorem describes conditions of attainment of scale invariance property by customer equivalent utility premium calculation principle.

4. *Customer equivalent utility premium calculation principle possesses scale invariance property if and only if* $u(x) = ax+b$, *for* $a > 0$, *i.e. only in the case when it coincides with net premium principle.*

Proof. We begin again from the sufficiency.

From equation (1), for linear utility function $u(x) = ax + b$ with $a > 0$, any risk X, and any customer's initial capital ω, follows

$$a\omega - a\pi_{\text{c.e.u.}}[X] + b \;=\; \mathsf{E}[a\omega - aX + b] \;=\; a\omega - a\mathsf{E}[X] + b,$$

hence

$$\pi_{\text{c.e.u.}}[X] \;=\; \mathsf{E}[X] \;=\; \pi_{\text{net}}[X].$$

On the other hand, from equation (1), for any $\Theta > 0$, the same risk X, the same customer's initial capital ω, and the same customer's utility function, follows

$$a\omega - a\pi_{\text{c.e.u.}}[\Theta X] + b \;=\; \mathsf{E}[a\omega - a\Theta X + b] \;=\; a\omega - a\Theta\mathsf{E}[X] + b,$$

thus

$$\pi_{\text{c.e.u.}}[\Theta X] \;=\; \Theta\mathsf{E}[X] \;=\; \Theta\pi_{\text{c.e.u.}}[X],$$

and we see that customer equivalent utility premium calculation principle possesses scale invariance property in the case of linear customer's utility function.

Proof of the sufficiency was finished, so we can start to prove the necessity.

To show that customer equivalent utility premium calculation principle with non-linear customer utility function $u(x)$ will not possess scale invariance property, we will choose risk X which takes only two possible values, namely 0 and t (here t is a non-zero real constant) with probabilities $1-p$ and p respectively. Risk X can in this case be considered as a random function of two parameters, namely p and t, and, therefore, within the proof of Theorem 4 will be denoted by X_p^t.

For any customer's initial capital ω, and any admissible customer's utility function $u(x)$, customer's equivalent utility equation (1) for risk X_p^t will take a form

$$u(\omega - \pi_{\text{c.e.u.}}[X_p^t]) \;=\; u(\omega - t)\cdot p + u(\omega)\cdot(1-p). \tag{56}$$

Substituting $p = 0$ into equation (56), obtain

$$u(\omega - \pi_{\text{c.e.u.}}[X_0^t]) \;=\; u(\omega). \tag{57}$$

Since $u(x)$ is strictly increasing function, then equation (57) yields

$$\pi_{\text{c.e.u.}}[X_0^t] \;=\; 0. \tag{58}$$

Let us now take partial derivative with respect to p from both sides of equation (56)

$$-u'(\omega - \pi_{\text{c.e.u.}}[X_p^t])\cdot\frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t] \;=\; u(\omega - t) - u(\omega). \tag{59}$$

Substituting $p = 0$ into equation (59) and using identity (58), obtain

$$-u'(\omega)\cdot\frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t]\bigg|_{p=0} \;=\; u(\omega - t) - u(\omega). \tag{60}$$

In the case of scale invariant customer equivalent utility premium calculation principle for any $\Theta > 0$ must hold identity

$$\pi_{\text{c.e.u.}}[\Theta X_p^t] = \Theta\pi_{\text{c.e.u.}}[X_p^t],$$

therefore, in the case when customer equivalent utility premium calculation principle possesses scale invariance property, equation (1) for risk ΘX_p^t will have a form

$$u(\omega - \Theta\pi_{\text{c.e.u.}}[X_p^t]) \;=\; u(\omega - \Theta t)\cdot p + u(\omega)\cdot(1-p). \tag{61}$$

We are now taking partial derivative with respect to p from both sides of equation (61)

$$-u'(\omega - \pi_{\text{c.e.u.}}[X_p^t])\cdot\Theta\cdot\frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t] \;=\; u(\omega - \Theta t) - u(\omega). \tag{62}$$

Substituting $p = 0$ into equation (62), and using identity (58), obtain

$$-u'(\omega)\cdot\Theta\cdot\frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t]\bigg|_{p=0} \;=\; u(\omega - \Theta t) - u(\omega). \tag{63}$$

Since $\Theta > 0$, then, with out of loss of generality, equation (63) can be rewritten as follows

$$-u'(\omega)\cdot\frac{\partial}{\partial p}\pi_{\text{c.e.u.}}[X_p^t]\bigg|_{p=0} \;=\; \frac{u(\omega - \Theta t) - u(\omega)}{\Theta}. \tag{64}$$

Observe that equations (60) and (64) have equal left hand sides, this means that their right hand sides also have to be equal, and in this way we finally get an equation which customer utility function $u(x)$ has to satisfy for premium calculation principle to be scale invariant, namely

$$u(\omega - t) - u(\omega) \;=\; \frac{u(\omega - \Theta t) - u(\omega)}{\Theta}. \tag{65}$$

Taking partial derivatives with respect to parameter t from both sides of (65), get

$$u'(\omega - t) \;=\; u'(\omega - \Theta t). \tag{66}$$

By fixing values of parameters ω and t, and changing values of parameter Θ, we will make $u'(\omega - \Theta t)$ a function of changing variable while value $u'(\omega - t)$ will be a fixed constant. Using this technique and taking into account monotonicity of function $u(\cdot)$ and continuity of function $u'(\cdot)$, since $u(\cdot) \in C_2(\mathbb{R})$, using equation (66) we conclude that

$$u'(x) \;=\; a > 0, \quad \text{for } x \in \mathbb{R}.$$

Integrating yields

$$u(x) \;=\; ax + b, \quad \text{for } x \in \mathbb{R}, \;\; \text{and constant } a > 0.$$

Let us give also a geometrical interpretation showing that non-linear customer utility functions will not satisfy equation (65). Let us consider two triangles: the first one will be formed by points $(\omega - t, u(\omega - t))$, $(\omega, u(\omega - t))$, $(\omega, u(\omega))$ and the second one will be formed by points $(\omega - \Theta t, u(\omega - \Theta t))$, $(\omega, u(\omega - \Theta t))$, $(\omega, u(\omega))$. Observe that both triangles are right-angled triangles, they have a common vertex at point $(\omega, u(\omega))$, and, moreover, points $(\omega, u(\omega))$, $(\omega, u(\omega - t))$, and $(\omega, u(\omega - \Theta t))$ lie on the same straight line. Since $t \in \mathbb{R} \setminus \{0\}$ then, with out of loss of generality, equation (65) can be rewritten as

$$\frac{u(\omega) - u(\omega - t)}{\omega - (\omega - t)} \;=\; \frac{u(\omega) - u(\omega - \Theta t)}{\omega - (\omega - \Theta t)}. \tag{67}$$

Geometrically, equation (67) can be interpreted as follows: ratio of cathetuses in one of the triangles is equal to ratio of cathetuses in the other triangle, hence our two considered triangles are similar triangles. Due to the common vertex, cathetuses which lie on a common straight line, and vertexes which lie on the same half plane with respect to the mentioned line, we conclude that hypotenuses will also lie on a common straight line; in other words, points $(\omega - \Theta t, u(\omega - \Theta t))$, for any initial capital ω, any non-zero t, and every $\Theta > 0$, will form a straight line. So, we can conclude that customer's utility function $u(x)$ is linear function, i.e. function of the form $u(x) = ax + b$. Initial assumption of positivity of first derivative of function $u(x)$ gives us additional restriction on parameter a: parameter a must be a strictly positive constant. This completes the proof of Theorem 4. □

Applying contradiction technique, proof of Theorem 4 can be used for showing that case of $u(x) = ax + b$, for $a > 0$, is the only case when customer equivalent utility premium principle coincides with net premium principle. Indeed, let us assume that for some function $u(x)$, different from linear function, customer equivalent utility principle will be equivalent to net premium principle. Then, due to linearity property of expectation, such method of pricing must be scale invariant. However it was shown in the proof of

Theorem 4 that the only case when customer equivalent utility principle is scale invariant is the case of $u(x) = ax + b$, for $a > 0$, so we come to a contradiction.

Due to arbitrary choice of customer's initial capital in the proof of Theorem 4 and no restrictions on it within the proof, the following corollary to Theorem 4 can be formulated.

4. *Customer zero utility premium calculation principle possesses scale invariance property if and only if $u(x) = ax + b$, for $a > 0$, i.e. only in the case when it coincides with net premium principle.*

As was already mentioned, in the case when customer zero utility premium calculation principle is applied to a special class of risks, it is enough to define utility function $u(x)$ on a subset $A \subset \mathbb{R}$ preserving monotonicity and concavity properties, i.e. $u(x)$ must be such that $u'(x) > 0$ and $u''(x) \leq 0$ for all $x \in A$, and, moreover, equation (2) must preserve its correct mathematical meaning for all risks from the mentioned class. It is interesting to see that in the case of subjecting of customer zero utility premium calculation principle to pricing of only strictly positive risks class of functions $u(x)$ producing scale invariant premiums is larger than in the general case. We believe that this observation deserves to be formulated in a form of theorem.

The following theorem is valid only for customer zero utility principle and not for customer equivalent utility principle.

5. *Customer zero utility premium calculation principle subjected to consideration of only strictly positive risks possesses scale invariance property if and only if $u(x) = -a(-x)^{\kappa} + b$, for $a > 0$ and $\kappa \geq 1$, defined for $x \in (-\infty, 0)$.*

Observe that for function $u(x) = -a(-x)^{\kappa} + b$ with $a > 0$ and $\kappa > 1$ condition $u'(x) > 0$ violates at point $x = 0$, therefore, statement of Theorem 5 does not contradict statement of Theorem 4.

Proof. Since in the case of strictly positive risk X we get $\mathsf{E}[X] > 0$, then, combining Jensen inequality

$$u(-\mathsf{E}[X]) \;\geq\; \mathsf{E}[u(-X)]$$

with definition equation (2), we see that customer zero utility premium calculation principle will be well-defined if function $u(x)$ will be defined just for $x \in (-\infty, 0)$ with preservation of monotonicity and concavity assumptions i.e. function $u(x)$ must be defined on $(-\infty, 0)$ such that $u'(x) > 0$ and $u''(x) \leq 0$ for all $x \in (-\infty, 0)$.

Let us from the beginning prove sufficiency of the statement. Indeed in the case of $u(x) = -a(-x)^{\kappa} + b$, with $a > 0$ and $\kappa \geq 1$, for any strictly positive risk X equation (2) will have a form

$$-a(\pi_{\text{c.z.u.}}[X])^{\kappa} + b \;=\; \mathsf{E}[-aX^{\kappa} + b] \;=\; -a\mathsf{E}[X^{\kappa}] + b,$$

therefore, in the considered case

$$\pi_{\text{c.z.u.}}[X] \;=\; (\mathsf{E}[X^\kappa])^{1/\kappa}.$$

On the other hand, for the same function $u(x)$, the same risk X, and any $\Theta > 0$, from equation (2) follows

$$-a(\pi_{\text{c.z.u.}}[\Theta X])^\kappa + b \;=\; \mathsf{E}[-a(\Theta X)^\kappa + b] \;=\; -a\Theta^\kappa \mathsf{E}[X^\kappa] + b$$

so, here we get

$$\pi_{\text{c.z.u.}}[\Theta X] \;=\; \Theta(\mathsf{E}[X^\kappa])^{1/\kappa} \;=\; \Theta\pi_{\text{c.z.u.}}[X],$$

and as we see, customer zero utility premium calculation principle subjected to consideration of only strictly positive risks possesses scale invariance property in the case of $u(x) = -a(-x)^\kappa + b$, for $a > 0$ and $\kappa \geq 1$, defined for $x \in (-\infty, 0)$.

Let us now switch to the statement of necessity. In order to show that customer zero utility premium calculation principle subjected to consideration of only strictly positive risks with all other types of function $u(x)$ will not possess scale invariance property, we will consider risk X taking values $\varepsilon > 0$ and 1 with probabilities p and $1-p$ respectively. Being a random function of parameters ε and p, risk X within the proof of Theorem 5 will be denoted as X_p^ε.

For described risk X_p^ε equation (2) will have a form

$$u(-\pi_{\text{c.z.u.}}[X_p^\varepsilon]) \;=\; u(-\varepsilon)\cdot p + u(-1)\cdot(1-p). \tag{68}$$

From equation (68) follows

$$u(-\pi_{\text{c.z.u.}}[X_0^\varepsilon]) \;=\; u(-1),$$

moreover, since $u(x)$ is a strictly increasing function, then

$$\pi_{\text{c.z.u.}}[X_0^\varepsilon] \;=\; 1. \tag{69}$$

Calculating partial derivatives with respect to parameter p from both sides of equation (68), obtain

$$-u'(-\pi_{\text{c.z.u.}}[X_p^\varepsilon])\cdot\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon] \;=\; u(-\varepsilon) - u(-1). \tag{70}$$

Substituting $p = 0$ into equation (70), obtain

$$-u'(-\pi_{\text{c.z.u.}}[X_0^\varepsilon])\cdot\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0} \;=\; u(-\varepsilon) - u(-1). \tag{71}$$

Using (69) equation (71) can be rewritten as

$$-u'(-1)\cdot\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0} \;=\; u(-\varepsilon) - u(-1). \tag{72}$$

Let us now calculate partial derivatives with respect to parameter p from both sides of equation (70)

$$u''(-\pi_{\text{c.z.u.}}[X_p^\varepsilon]) \cdot \left(\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\right)^2 - u'(-\pi_{\text{c.z.u.}}[X_p^\varepsilon]) \cdot \frac{\partial^2}{(\partial p)^2}\pi_{\text{c.z.u.}}[X_p^\varepsilon] \;=\; 0. \tag{73}$$

Substituting $p = 0$ into equation (73), and using identity (69), obtain

$$u''(-1) \cdot \left(\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right)^2 - u'(-1) \cdot \left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right) \;=\; 0. \tag{74}$$

Taking ε small enough, namely $\varepsilon < 1$, and taking into account strict monotonicity of function $u(x)$, with out of loss of generality, using (72), we may conclude that

$$\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0} \;\neq\; 0, \tag{75}$$

hence, equation (74) can be rewritten as

$$\frac{u''(-1)}{u'(-1)} \;=\; \left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right) \bigg/ \left(\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right)^2. \tag{76}$$

For any $\Theta > 0$, equation (2) for risk ΘX_p^ε will take a form

$$u(-\pi_{\text{c.z.u.}}[\Theta X_p^\varepsilon]) \;=\; u(-\Theta\varepsilon) \cdot p + u(-\Theta) \cdot (1-p). \tag{77}$$

In the case of scale invariant customer zero utility premium principle equation (77) can be rewritten as

$$u(-\Theta\pi_{\text{c.z.u.}}[X_p^\varepsilon]) \;=\; u(-\Theta\varepsilon) \cdot p + u(-\Theta) \cdot (1-p). \tag{78}$$

Calculating second partial derivative with respect to p from both sides of equation (78), obtain

$$\begin{aligned} &u''(-\Theta\pi_{\text{c.z.u.}}[X_p^\varepsilon]) \cdot \Theta^2 \cdot \left(\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\right)^2 - \\ &\qquad - \; u'(-\Theta\pi_{\text{c.z.u.}}[X_p^\varepsilon]) \cdot \Theta \cdot \frac{\partial^2}{(\partial p)^2}\pi_{\text{c.z.u.}}[X_p^\varepsilon] \;=\; 0. \end{aligned} \tag{79}$$

Substituting $p = 0$ into equation (79), canceling Θ factor, and using identity (69), get

$$u''(-\Theta) \cdot \Theta \cdot \left(\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right)^2 - u'(-\Theta) \cdot \left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right) \;=\; 0. \tag{80}$$

Since $u'(-\Theta) > 0$, then using relation (75), equation (80) can be rewritten as

$$\frac{u''(-\Theta) \cdot \Theta}{u'(-\Theta)} \;=\; \left(\frac{\partial^2}{(\partial p)^2}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right) \bigg/ \left(\frac{\partial}{\partial p}\pi_{\text{c.z.u.}}[X_p^\varepsilon]\bigg|_{p=0}\right)^2. \tag{81}$$

Observe that equations (76) and (81) have equal right hand sides, this means that their left hand sides also have to be equal, in this way we finally get an equation which function $u(x)$ has to satisfy in the case of scale invariant customer zero utility premium calculation principle subjected to consideration of only strictly positive risks, namely,

$$\frac{u''(-\Theta)\cdot\Theta}{u'(-\Theta)} = \frac{u''(-1)}{u'(-1)}, \quad \text{for all } \Theta > 0. \tag{82}$$

Assigning $-u''(-1)/u'(-1) =: \varkappa$ (since $u''(-1) \leq 0$ and $u'(-1) > 0$ then $\varkappa \geq 0$) and making substitution $z(\Theta) := u'(-\Theta)$ equation (82) can be rewritten in the following equivalent form

$$\frac{dz}{z} = \varkappa\frac{d\Theta}{\Theta},$$

therefore

$$\log(z(\Theta)) = \varkappa\log(\Theta) + \log(C_1), \quad \text{for some constant } C_1 > 0,$$

and function $z(\Theta)$ itself will have a form

$$z(\Theta) = C_1\Theta^{\varkappa}.$$

Switching back to function $u'(-\Theta)$, obtain

$$u'(-\Theta) = C_1\Theta^{\varkappa}. \tag{83}$$

Switching back to the original parameter $x \in (-\infty, 0)$ representation (83) can be rewritten as

$$u'(x) = C_1(-x)^{\varkappa}.$$

Taking antiderivative, obtain

$$u(x) = -\frac{C_1}{\varkappa+1}(-x)^{\varkappa+1} + C_2,$$

therefore function $u(x)$ must be a function of the form

$$u(x) = -a(-x)^{\kappa} + b, \quad \text{for some real constants } a,\ b, \text{ and } \kappa.$$

Moreover, since $C_1 > 0$ and $\varkappa > 0$ then $a > 0$, and since $\varkappa \geq 0$ then $\kappa \geq 1$.

This completes the proof of Theorem 5. □

References

[1] *Asmussen S., Albrecher H.* Ruin Probabilities (second edition). World Sientific, Singapore, 2010.

[2] *Boland P.J.* Statistical and Probabilistic Methods in Actuarial Science. Chapman & Hall, Boca Raton, 2007.

[3] *Bowers N.L., Gerber H.U., Hickman J.C., Jones D.A., Nesbit C.J.* Actuarial Mathematics (second edition). The Society of Actuaries, Illinoice, 1997.

[4] *Bühlmann H.* Mathematical Methods in Risk Theory. Springer, Berlin, 1970.

[5] *Dickson D.C.M.* Insurance Risk and Ruin. Cambridge University Press, Cambridge, 2005.

[6] *Gerber H.U.* An Introduction to Mathematical Risk Theory. S. S. Huebner Foundation for Insurance Education, Philadelpia, 1979.

[7] *De Vylder F.E., Goovaerts M., Haezendonck J.* (*editors*) Premium Calculation in Insurance (collection of articles). Kluwer Academic Publishers, Boston, 1984.

[8] *De Vylder F.E., Goovaerts M., Haezendonck J.* (*editors*) Insurance and Risk Theory (collection of articles). Kluwer Academic Publishers, Boston, 1986.

[9] *Kaas R., Goovaerts M., Dhaene J., Denuit M.* Modern Actuarial Risk Theory using R. Springer, Berlin, 2008.

[10] *Kremer E.* Applied Risk Theory. Shaker, Aachen, 1999.

[11] *Rolski T., Schmidli H., Schmidt V., Teugels J.* Stochastic Processes for Insurance and Finance. John Wiley & Sons, Chichester, 1999.

[12] *Straub E.* Non-Life Insurance Mathematics. Springer, Berlin, 1988.