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On one class of functions related to Ostrogradsky series and containing singular and nowhere monotonic functions

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Abstract. In the paper, we consider the infinite system of functional equations depending on sequence of parameters (p_n) such that $|p_n| < 1$, $\sum\limits_{n=1}^{\infty} p_n = 1$. The solution of this system is the continuous function defined at irrational point of $(0,1)$ by equality

$$F(x) = F(\bar{\text{O}}^1(g_1(x), g_2(x), \ldots, g_n(x), \ldots)) = \beta_{g_1(x)} + \sum_{n\geq 2} (-1)^{n-1} \beta_{g_n(x)} \prod_{i=1}^{n-1} p_{g_i(x)},$$

where $\beta_1 = 1$, $\beta_{n+1} = 1 - \sum\limits_{i=1}^{n} p_i > 0$, and $\bar{\text{O}}^1(g_1(x), g_2(x), \ldots, g_n(x), \ldots)$ is a formal expression of number x by alternating first Ostrogradsky series, i.e.,

$$x = \frac{1}{q_1(x)} - \frac{1}{q_1(x)q_2(x)} + \ldots + \frac{(-1)^{n-1}}{q_1(x)q_2(x)\ldots q_n(x)} + \ldots,$$

$g_1(x) = q_1(x)$, $g_n(x) = q_{n+1}(x) - q_n(x)$.

We study structural, differential, fractal properties of function F according to the sequence (p_n). “Most” of such functions are singular and nowhere monotonic, and singular non-monotonic functions form an essential class of them. We prove that function is nowhere monotonic if the sequence (p_n) does not have zeroes but has negative terms.

Keywords: first Ostrogradsky series, representation of real number, infinite system of functional equations, singular function, nowhere monotonic function, Lebesgue measure, fractal Hausdorff–Besicovitch dimension

Анотація. Об'єктом дослідження даної роботи є нескінченна система функціональних рівнянь, залежна від послідовності параметрів (p_n) такої, що $|p_n| < 1$, $\sum\limits_{n=1}^{\infty} p_n = 1$, розв'язком якої є неперервна функція, визначена в ірраціональній точці $(0,1)$ рівністю

$$F(x) = F(\bar{\mathrm{O}}^1(g_1(x), g_2(x), \ldots, g_n(x), \ldots)) = \beta_{g_1(x)} + \sum_{n\geq 2} (-1)^{n-1} \beta_{g_n(x)} \prod_{i=1}^{n-1} p_{g_i(x)},$$

де $\beta_1 = 1$, $\beta_{n+1} = 1 - \sum\limits_{i=1}^{n} p_i > 0$, а $\bar{\mathrm{O}}^1(g_1(x), g_2(x), \ldots, g_n(x), \ldots)$ — це формальний запис числа x знакозмінним рядом Остроградського 1-го виду, тобто

$$x = \frac{1}{q_1(x)} - \frac{1}{q_1(x)q_2(x)} + \ldots + \frac{(-1)^{n-1}}{q_1(x)q_2(x)\ldots q_n(x)} + \ldots,$$

$g_1(x) = q_1(x)$, $g_n(x) = q_{n+1}(x) - q_n(x)$.

Вивчаються структурні, диференціальні, фрактальні властивості функції F в залежності від послідовності (p_n). «Більшість» таких функцій є сингулярними та ніде не монотонними, серед них істотний клас утворюють сингулярні немонотонні функції. Доведено, що функція є ніде не монотонною, якщо послідовність (p_n) не має нулів, але має від'ємні члени.

Ключові слова: ряд Остроградського 1-го виду, зображення дійсного числа, нескінченна система функціональних рівнянь, сингулярна функція, ніде не монотонна функція, міра Лебега, фрактальна розмірність Хаусдорфа–Безиковича

Introduction

One of the many models for the general axiomatic theory of real numbers is the representation of real numbers by the first Ostrogradsky series [18, 8, 14, 16, 9, 22] (also known as Pierce series [20, 21, 11, 12, 27]):

$$q_0 + \frac{1}{q_1} - \frac{1}{q_1 q_2} + \ldots + \frac{(-1)^{k-1}}{q_1 q_2 \cdots q_k} + \ldots, \tag{1}$$

where $q_0 \in \mathbb{Z}$, $q_k \in \mathbb{N}$ and $q_{k+1} > q_k$. Any irrational number from $(0, 1]$ can be uniquely represented in the form of series (1) with $q_0 = 0$, and rational numbers have two such representations, both of which are finite.

Expression (1) for a real number $x \in (0, 1]$ can be represented in the following form:

$$x = \frac{1}{g_1} - \frac{1}{g_1(g_1+g_2)} + \cdots + \frac{(-1)^{n-1}}{g_1(g_1+g_2)\ldots(g_1+\cdots+g_n)} + \cdots \tag{2}$$

$$\equiv \bar{\mathrm{O}}^1(g_1, g_2, \ldots, g_n, \ldots), \tag{3}$$

where $g_1 = q_1$, $g_n = q_{n+1} - q_n$. The expression (2) is said to be the $\bar{\mathrm{O}}^1$-*representation* of the number x and the number $g_n = g_n(x)$ is said to be its nth $\bar{\mathrm{O}}^1$-*symbol*. The advantage of the expression (2) over (1) is that digits ($\bar{\mathrm{O}}^1$-symbols) of the alphabet $\mathbb{N}$ in the representation of a number are "peer" (unlike numbers of the sequence (q_k) such that $q_{k+1} > q_k$).

The history of evolution of the theory for this representation and a brief review of literature can be found in [11, 12, 9, 22].

This representation has the following features:

(1) As expansions of numbers to regular continued fractions it has an infinite alphabet unlike the s-adic representation or the Q^*-representation [1, 14].
(2) The denominators in the series (1) are products of positive integers (i.e., a real number is modeled by positive integers).
(3) Series (1) converges rapidly. This is important for approximation theory.
(4) The system has a zero redundancy: an irrational number has a unique representation, and a rational number has two representations, both are finite.
(5) Topological and geometric properties of this representation are analogous to properties of the representation of real numbers by the regular continued fractions whereas its metric theory is essentially different [9].
(6) The metric theory of this representation has much in common with the metric theory of the representation of real numbers by Engel series [17].
(7) The "geometry" of this representation is not "self-similar" in the classic sense, but has some features of "topological and metric N-self-similarity" [14].

Diversity of systems of representations of real numbers is a powerful tool for the modeling and study of objects of continuous mathematics with complicated local structure. First of all, this paper is devoted to functions without monotonicity intervals (in particular, non-differentiable functions) and singular probability distribution functions.

In previous papers we studied the geometry of the $\bar{\mathrm{O}}^1$-representation (geometric meaning of $\bar{\mathrm{O}}^1$-symbols, properties of cylindrical sets, metric relations) [9], topological, metric and fractal properties of sets with conditions on digits [9, 2] as well as properties of the random variable

$$\xi = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{\eta_1(\eta_1+\eta_2)\dots(\eta_1+\eta_2+\dots+\eta_k)}, \tag{4}$$

where the $\bar{\mathrm{O}}^1$-symbols η_k are independent random variables with distributions $\mathsf{P}\{\eta_k = i\} = p_{ik} \geq 0$, $\sum\limits_{i\in\mathbb{N}} p_{ik} = 1$, $i \in \mathbb{N}$, $k \in \mathbb{N}$ [16, 2]. However, even for identically distributed η_k ($p_{ik} = p_i$ for any $k \in \mathbb{N}$) we did not obtain necessary and sufficient conditions for ξ to be of Cantor type (i.e., with its spectrum continuum and having zero Lebesgue measure). In the present paper we return to these problems and prove new results about metric properties of sets of numbers such that their $\bar{\mathrm{O}}^1$-representation uses only part of the alphabet. These sets are the spectra for the random variables under consideration.

There are some common problems in the theory of functions with complicated local structure (singular, nowhere monotonic, crinkly, non-differentiable), in particular, the problem of developing effective tools for their representation and methods for their study. In this paper we use systems of functional equations. In general, this approach is not so widely used in probability distributions theory.

First of all, we give an equivalent definition of the probability distribution function of a random variable ξ in the form of a unique solution of an infinite system of functional equations in the class of bounded functions defined on $[0,1]$.

We proved in previous papers [16] that the spectrum of ξ is the closure of the set

$$C \equiv C[\bar{\text{O}}^1, V] = \left\{x : x = \bar{\text{O}}^1(g_1, g_2, \dots, g_n, \dots),\ \mathsf{P}\{\eta_k = g_k\} > 0,\ g_k \in \mathbb{N}\right\},$$

and the set C is of zero Lebesgue measure if the set V of "allowed" symbols does not contain arbitrarily long sequences of sequential symbols. Now we prove that this condition is not necessary. We give two families of Cantor type probability distributions without this condition.

1. Main object: system of functional equations

Let (p_i) be a sequence of real numbers with the following properties

(1) $\sum\limits_{i=1}^{\infty} p_i = 1$,

(2) $|p_i| < 1$ for all $i \in \mathbb{N}$,

(3) $\beta_{k+1} = 1 - \sum\limits_{i=1}^{k} p_i > 0$ for all $k \in \mathbb{N}$ and $\beta_1 = 1$.

We consider the system of functional equations

$$\begin{cases} f(\bar{\text{O}}^1(i)) = \beta_i, \qquad i \in \mathbb{N}, \\ f(\bar{\text{O}}^1(i, g_1, g_2, \dots, g_n)) = \beta_i - p_i f(\bar{\text{O}}^1(g_1, g_2, \dots, g_n)), \\ f(\bar{\text{O}}^1(i, g_1, g_2, \dots, g_n, \dots)) = \beta_i - p_i f(\bar{\text{O}}^1(g_1, g_2, \dots, g_n, \dots)). \end{cases} \tag{5}$$

We are interesting in solutions of this system which are bounded functions f, defined on $(0,1]$.

Note that the system (5) is well-defined, that is the value of the function f is the same for different representations of any rational number x. In fact, for any $m \in \mathbb{N}$,

$$f(\bar{\text{O}}^1(g_1, g_2, \dots, g_{m-1}, g_m, 1)) = \sum_{k=1}^{m-1} (-1)^{k-1} \beta_{g_k} \prod_{i=1}^{k-1} p_{g_i} + (-1)^{m-1} \prod_{i=1}^{m-1} p_{g_i} f(\bar{\text{O}}^1(g_m, 1)),$$

$$f(\bar{\text{O}}^1(g_1, g_2, \dots, g_{m-1}, g_m + 1)) = \sum_{k=1}^{m-1} (-1)^{k-1} \beta_{g_k} \prod_{i=1}^{k-1} p_{g_i} + (-1)^{m-1} \prod_{i=1}^{m-1} p_{g_i} f(\bar{\text{O}}^1(g_m + 1)),$$

and

$$\begin{aligned} f(\bar{\mathrm{O}}^1(g_m,1)) &= \beta_{g_m} - p_{g_m} f(\bar{\mathrm{O}}^1(1)) \\ &= \beta_{g_m} - p_{g_m}\beta_1 = \beta_{g_m+1} = f(\bar{\mathrm{O}}^1(g_m+1)). \end{aligned}$$

2. Solution of the system and its form

1. *If a function F is defined on $(0,1]$, is bounded and satisfies the system (5) then the following expansion holds:*

$$F(x) = \beta_{g_1} + \sum_{k\geq 2} (-1)^{k-1} \beta_{g_k} \prod_{i=1}^{k-1} p_{g_i}, \tag{6}$$

where $g_k = g_k(x)$ is the kth $\bar{\mathrm{O}}^1$-symbol of x, and the sum over k contains infinitely many terms if x is irrational and finitely many otherwise.

Proof. Using the method of mathematical induction, we obtain

$$\begin{aligned} F(x) = F(\bar{\mathrm{O}}^1(g_1,g_2,\ldots,g_k,\ldots)) &= \sum_{k=1}^{m} (-1)^{k-1}\beta_{g_k}\prod_{i=1}^{k-1} p_{g_i} \\ &+ (-1)^m \prod_{i=1}^{m} p_{g_i} F(\bar{\mathrm{O}}^1(g_{m+1},g_{m+2},\ldots,g_k,\ldots)). \end{aligned} \tag{7}$$

Since $|p_i| < 1$ and therefore

$$\Big|\prod_{i=1}^{m} p_{g_i}\Big| \leq (\max_{i\in\mathbb{N}} \{|p_i|\})^m \to 0 \quad (m\to\infty),$$

and F is a bounded function, the remainder term in the expression (7) tends to 0 as $m\to\infty$. Hence, equality (6) holds for irrational x.

We obtain a finite expression (6) for any rational number

$$x = \bar{\mathrm{O}}^1(g_1,g_2,\ldots,g_m)$$

by iterating $m-1$ times the second equation of (5). □

1. *The system of functional equations (5) has a unique solution (6) in class of bounded functions defined on $(0,1]$.*

This corollary follows from facts that the $\bar{\mathrm{O}}^1$-representation has a zero redundancy and the right side of (6) gives the same results for different representations of a rational number x.

For study properties of the function (6), let us recall some notions and metric relations from the theory of $\bar{\mathrm{O}}^1$-representation of numbers which we shall need henceforth.

Let c_1, c_2, …, c_m be a fixed sequence of positive integers.

1. The set $\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m]}$, which is the closure of the set of all numbers $x \in (0,1)$, whose first m $\bar{\mathrm{O}}^1$-symbols are equal to c_1, c_2, $\ldots$, c_m respectively, is said to be *the cylindrical set (cylinder) of rank m with the base $c_1c_2\ldots c_m$*, i.e.,

$$\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m]} = \big(\{x : x = \bar{\mathrm{O}}^1(g_1(x), \ldots, g_n(x), \ldots), g_k(x) = c_k, 1 \le k \le m\}\big).$$

It is not hard to prove that a cylindrical set $\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m]}$ is a closed interval whose length is given by

$$\left|\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m]}\right| = \frac{1}{\sigma_1\sigma_2\ldots\sigma_m(\sigma_m+1)}, \tag{8}$$

where $\sigma_k = \sum\limits_{i=1}^{k} c_i$.

1. *We shall denote by $\bar{\mathrm{O}}^1_{(c_1c_2\ldots c_m)}$ the interior part of the set $\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m]}$, i.e., interval with the same endpoints as cylinder $\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m]}$.*

1 (important metric relations)**.** *For any positive integer s and m-tuple $(c_1, c_2, \ldots, c_m)$ of positive integers, the following equalities hold:*

$$\frac{\left|\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m s]}\right|}{\left|\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m]}\right|} = \frac{\sigma_m+1}{(\sigma_m+s)(\sigma_m+s+1)}, \tag{9}$$

$$\left|\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m s1]}\right| = \frac{1}{\sigma_m+s}\left|\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m(s+1)]}\right|, \tag{10}$$

$$\left|\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m s]}\right| = \frac{1}{\sigma_m+s}\sum_{j=s+1}^{\infty}\left|\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m j]}\right|. \tag{11}$$

Proof. Equalities (9) and (10) follows immediately from (8). Equality (11) follows from (8) and

$$\begin{aligned}\sum_{j=s+1}^{\infty}\left|\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m j]}\right| &= \sum_{j=s+1}^{\infty}\frac{1}{\sigma_1\sigma_2\ldots\sigma_m(\sigma_m+j)(\sigma_m+j+1)}\\ &= \frac{1}{\sigma_1\sigma_2\ldots\sigma_m}\sum_{j=s+1}^{\infty}\frac{1}{(\sigma_m+j)(\sigma_m+j+1)}\\ &= \frac{1}{\sigma_1\sigma_2\ldots\sigma_m(\sigma_m+s+1)}.\end{aligned}$$

□

2. *For any positive integer s and m-tuple $(c_1, c_2, \ldots, c_m)$ of positive integers, the following inequality holds:*

$$\frac{\left|\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m s]}\right|}{\left|\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m]}\right|} \le \frac{1}{2\cdot(2s-1)}. \tag{12}$$

Moreover, for $m \geq s-1$:

$$\frac{\left|\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m s]}\right|}{\left|\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_m]}\right|} \leq \frac{m+1}{(m+s)(m+s+1)}. \tag{13}$$

2. *The function F is continuous on the interval $(0,1)$, and left-continuous in a point $x=1$.*

Proof. For any $x_0 \in (0,1]$ from (7) it follows that

$$F(x) - F(x_0) = (-1)^m \prod_{i=1}^{m} p_{g_i} \Big(F(\bar{\mathrm{O}}^1(g_{m+1}(x), \ldots, g_k(x), \ldots)) - F(\bar{\mathrm{O}}^1(g_{m+1}(x_0), \ldots, g_k(x_0), \ldots)) \Big),$$

where m is a positive integer such that $g_i(x) = g_i(x_0)$ for all $i \leq m$ and $g_{m+1}(x) \neq g_{m+1}(x_0)$.

If x_0 is an irrational number then $x \to x_0$ if and only if $m \to \infty$ and

$$|F(x) - F(x_0)| \leq C\left|\prod_{i=1}^{m} p_{g_i}\right| \to 0 \quad (m \to \infty),$$

that is $\lim\limits_{x \to x_0} F(x) = F(x_0)$. □

2. *We can define the function F in the point $x=0$ due to continuity: $F(0)=0$, and obtain that F is defined and continuous on $[0,1]$.*

3. *The infinite system of functional equations*

$$f(\bar{\mathrm{O}}^1(i, g_1, g_2, \ldots, g_n, \ldots)) = \beta_i - p_i f(\bar{\mathrm{O}}^1(g_1, g_2, \ldots, g_n, \ldots)), \tag{14}$$

$i \in \mathbb{N}$, in the class of continuous functions on $[0,1]$ has a unique solution, namely the function (6).

3. Conditions for monotonicity and nowhere monotonicity of the function F

Let us define the change $\mu_F([a,b])$ in function F on a closed interval $[a,b]$ by the equality

$$\mu_F([a,b]) := F(b) - F(a).$$

3. *The change in the function F on a cylinder $\bar{\mathrm{O}}^1_{[c_1\ldots c_m]}$ is given by*

$$\mu_F(\bar{\mathrm{O}}^1_{[c_1\ldots c_m]}) = \prod_{i=1}^{m} p_{c_i}.$$

Proof. If m is odd number then

$$\begin{aligned}\mu_F(\bar{\text{O}}^1_{[c_1c_2\ldots c_m]}) &= F(\bar{\text{O}}^1(c_1,c_2,\ldots,c_m) - F(\bar{\text{O}}^1(c_1,c_2,\ldots,c_m+1)) \\ &= \left(F\left(\frac{1}{c_m}\right) - F\left(\frac{1}{c_m+1}\right)\right)\prod_{i=1}^{m-1} p_{c_i} = \prod_{i=1}^{m} p_{c_i}.\end{aligned}$$

If m is even number then

$$\begin{aligned}\mu_F(\bar{\text{O}}^1_{[c_1c_2\ldots c_m]}) &= F(\bar{\text{O}}^1(c_1,c_2,\ldots,c_m+1) - F(\bar{\text{O}}^1(c_1,c_2,\ldots,c_m)) \\ &= \left(F\left(\frac{1}{c_m+1}\right) - F\left(\frac{1}{c_m}\right)\right)\prod_{i=1}^{m-1} p_{c_i} = \prod_{i=1}^{m} p_{c_i}.\end{aligned}$$

□

3. *Let $(c_1,\ldots,c_m)$ be a given sequence of positive integers. If there exists $p_{c_k}=0$ with $k\le m$, then*

$$\mu_F(\bar{\text{O}}^1_{[c_1\ldots c_m]}) = 0.$$

2. *If $p_k\ne 0$ for any $k\in\mathbb{N}$ and there exist numbers p_k and p_j in the sequence (p_n) such that $p_kp_j<0$, then the function F does not have any arbitrary small monotonicity interval.*

Proof. Suppose that conditions of the theorem are fulfilled and let (a,b) be a monotonicity interval for the function F. Then there exists a cylinder $\bar{\text{O}}^1_{[c_1c_2\ldots c_m]}$ such that $\bar{\text{O}}^1_{[c_1c_2\ldots c_m]}\subset (a,b)$.

Since $p_k\ne 0$ for all $k\in\mathbb{N}$, using Lemma 3

$$\mu_F(\bar{\text{O}}^1_{[c_1c_2\ldots c_m]}) = \prod_{i=1}^{m} p_{c_i} \ne 0$$

and

$$\mu_F(\bar{\text{O}}^1_{[c_1c_2\ldots c_mk]})\cdot\mu_F(\bar{\text{O}}^1_{[c_1c_2\ldots c_mj]}) < 0,$$

that is the change in the function F is positive on one of the intervals $\bar{\text{O}}^1_{(c_1c_2\ldots c_mk)}$, $\bar{\text{O}}^1_{(c_1c_2\ldots c_mj)}$, and the change in the function F is negative on the other interval. This contradicts the monotonicity of the function F on the interval $\bar{\text{O}}^1_{(c_1c_2\ldots c_m)}$ and proves the theorem. □

4. *The function F is constant on the cylinder $\bar{\text{O}}^1_{[c_1\ldots c_m]}$ if and only if there exists $p_{c_k}=0$ with $k\le m$.*

4. *If $p_k\ge 0$ for any $k\in\mathbb{N}$ then the function F is a monotonic non-decreasing function on the closed interval $[0,1]$, moreover*

$$F(0) = \lim_{x\to 0+} F(x) = 0, \quad F(1) = 1.$$

Proof. Consider two irrational numbers $x_1, x_2 \in [0,1]$ such that $x_1 < x_2$. Then $g_i(x_1) = g_i(x_2)$ for all $i \leq m$ and $g_{m+1}(x_1) < g_{m+1}(x_2)$ for some m. We suppose here that m is an odd integer, in the case of an even m the proof is analogous.

For odd m from (7) it follows that

$$F(x_1) - F(x_2) = -\prod_{i=1}^{m} p_{g_i}\Big(F(\bar{\mathrm{O}}^1(g_{m+1}(x),\dots,g_k(x),\dots)) - F(\bar{\mathrm{O}}^1(g_{m+1}(x_0),\dots,g_k(x_0),\dots))\Big) \leq 0.$$

Moreover,

$$F(0) = \lim_{x\to 0+} F(x) = \lim_{x\to 0+} F\left(\frac{1}{n+1}\right) = \lim_{n\to\infty} \beta_{n+1} = 0,$$
$$F(1) = F\left(\frac{1}{1}\right) = \beta_1 = 1,$$

and the lemma is proved. □

5. *If $p_k \geq 0$ for any $k \in \mathbb{N}$ then the function F is a continuous probability distribution function on $[0,1]$.*

4. Differential properties of the function F

3. *If the function (6) has a derivative in an irrational point*

$$x_0 = \bar{\mathrm{O}}^1(g_1, g_2, \dots, g_n, \dots),$$

then

$$F'(x_0) = \lim_{m\to\infty} (\sigma_m + 1)\prod_{i=1}^{m} \sigma_i \left|p_{g_i(x)}\right|,$$

where $\sigma_i = g_1(x) + g_2(x) + \dots + g_i(x)$. If x_0 is a rational point, then $F'(x_0)$ does not exist.

Proof. Indeed, if the derivative exists in an irrational point x_0, then it is equal to

$$F'(x_0) = \lim_{\substack{x_n' < x_0 < x_n'' \\ x_n'' - x_n' \to 0}} \frac{F(x_n'') - F(x_n')}{x_n'' - x_n'},$$

where x_n' and x_n'' may be endpoints of cylinder containing point x_0. Then using expressions for length of cylinder and change in function we have

$$F'(x_0) = \lim_{m\to\infty} \frac{\left|\prod_{i=1}^{m} p_{g_i(x_0)}\right|}{\left|\bar{\mathrm{O}}^1_{[g_1(x_0)g_2(x_0)\dots g_m(x_0)]}\right|} = \lim_{m\to\infty} (\sigma_m + 1)\prod_{i=1}^{m} \sigma_i \left|p_{g_i(x_0)}\right|. \qquad \square$$

4. *If $p_k \geq 0$ for any $k \in \mathbb{N}$ then the function F can be interpreted as a singular probability distribution function of the random variable ξ with independent identically distributed $\bar{\mathrm{O}}^1$-symbols η_k taking the values $1, 2, \ldots, i, \ldots$ with probabilities $p_1, p_2, \ldots, p_i, \ldots$ respectively.*

Proof. It is known [16], that the distribution function of the random variable ξ is defined by the equality

$$F_\xi(x) = \beta_1(x) + \sum_{k\geq 2}(-1)^{k-1}\beta_k(x)\prod_{i=1}^{k-1} p_{g_i(x)i}, \quad \text{if } 0 < x \leq 1, \tag{15}$$

where

$$\beta_k(x) = 1 - \sum_{j=1}^{g_k(x)-1} p_{jk}$$

and $g_k(x)$ is the kth $\bar{\mathrm{O}}^1$-symbol of number x. The series in (15) is infinite if x is irrational and finite otherwise.

If $p_{mk} = p_m$ for all $k \in \mathbb{N}$ then we obtain (6) from expression (15). □

5. Cantor singular functions

2. The *spectrum* S_ζ of a probability distribution function F_ζ of a random variable ζ is the set of all point of increase of a function F_ζ, i.e.,

$$S_\zeta \equiv S_{F_\zeta} := \{x : F_\zeta(x+\varepsilon) - F_\zeta(x-\varepsilon) > 0 \; \forall \varepsilon > 0\} =$$
$$= \{x : \mathsf{P}\{\zeta \in (x-\varepsilon, x+\varepsilon)\} > 0 \; \forall \varepsilon > 0\}.$$

If S_ζ is a nowhere dense set of zero Lebesgue measure, then the continuous probability distribution ζ (and its probability distribution function F_ζ) is called *singular probability distribution* (*singular probability distribution function*) *of Cantor type.* If the probability distribution ζ is singularly continuous and S_ζ is a nowhere dense set of positive Lebesgue measure, then ζ (and the corresponding probability distribution function) is called *singular probability distribution* (*singular probability distribution function*) *of quasi-Cantor type* [14, . 69].

5. *If $p_k \geq 0$ for any $k \in \mathbb{N}$, then the spectrum of the function F is the closure of the set*

$$C = \left\{x : p_{g_k(x)} > 0 \forall k \in \mathbb{N}\right\} = \{x : g_i(x) \in V = \{i : p_i \neq 0\}\} \equiv C[\bar{\mathrm{O}}^1, V].$$

The topological properties of the Borel set $C[\bar{\mathrm{O}}^1, V]$ are well known [9]. An open problem is to find the Lebesgue measure of this set when V and $\bar{V} = \mathbb{N} \setminus V$ are infinite, moreover V is a "thin enough".

If the set V determining the spectrum ξ contains a subset

$$V_0 = \{m+1, m+2, \dots, m+k, \dots\},$$

where m is any given positive integer, then ξ has a quasi-Cantor-type probability distribution, since it is known that

$$\lambda(C[\bar{\text{O}}^1, V_0]) > \frac{1}{(m+1)^2},$$

see [9].

6. *The Lebesgue measure of the set* $C[\bar{\text{O}}^1, V]$ *is given by the formula*

$$\lambda(C[\bar{\text{O}}^1, V]) = \prod_{k=0}^{\infty} \left(1 - \frac{\lambda(\bar{F}_{k+1})}{\lambda(F_k)}\right), \tag{16}$$

where F_k *is a union of all cylinders of rank* k *such that their interior contains points belonging to the set* $C[\bar{\text{O}}^1, V]$, $F_0 = [0,1]$,

$$\bar{F}_{k+1} := F_k \setminus F_{k+1}.$$

Proof. It is easy to see that

$$\lambda(F_k) \leq \lambda(C[\bar{\text{O}}^1, V]) = \lim_{k\to\infty} \lambda(F_k).$$

Then

$$\lambda(C[\bar{\text{O}}^1, V]) = \lim_{k\to\infty} \left(\frac{\lambda(F_{k+1})}{\lambda(F_k)} \cdot \frac{\lambda(F_k)}{\lambda(F_{k-1})} \cdot \dots \cdot \frac{\lambda(F_1)}{\lambda(F_0)}\right) =$$
$$= \prod_{k=0}^{\infty} \frac{\lambda(F_{k+1})}{\lambda(F_k)} = \prod_{k=0}^{\infty} \frac{\lambda(F_k) - \lambda(\bar{F}_{k+1})}{\lambda(F_k)} =$$
$$= \prod_{k=0}^{\infty} \left(1 - \frac{\lambda(\bar{F}_{k+1})}{\lambda(F_k)}\right).$$

□

6. *The Lebesgue measure of the set* $C[\bar{\text{O}}^1, V]$ *is equal to* 0 *if and only if*

$$\sum_{k=1}^{\infty} \frac{\lambda(\bar{F}_{k+1})}{\lambda(F_k)} = \infty. \tag{17}$$

This corollary follows from known fact about relation between convergence of infinite products and series.

7. *If there exists positive constant* c *such that for all* k *large enough,*

$$\frac{\lambda(\bar{F}_{k+1})}{\lambda(F_k)} \geq c, \tag{18}$$

then the Lebesgue measure of the set $C[\bar{\text{O}}^1, V]$ *is equal to* 0.

7. *Let* $\bar{V} = \mathbb{N} \setminus V = \{u_1, u_2, \dots, u_n, \dots\}$, $u_n < u_{n+1}$. *If there exists a constant* $\gamma \in (0,1)$ *such that for all positive integers* $\sigma > \sigma_0$ *the following inequality holds:*

$$\varphi(\sigma) \equiv \sum_{n=1}^{\infty} \frac{1}{(\sigma + u_n)(\sigma + u_n + 1)} \geq \frac{\gamma}{\sigma + u_1}, \tag{19}$$

then the set $C[\bar{\text{O}}^1, V]$ *is of zero Lebesgue measure.*

Proof. Let us show that there exists a positive constant c such that the inequality (18) holds if conditions of the lemma are fulfilled. To this end we consider a cylinder $\bar{\text{O}}^1_{[c_1 \dots c_k]} \subset F_k$ and set

$$\bar{F}_{k+1} \cap \bar{\text{O}}^1_{[c_1 \dots c_k]} = \bigcup_{c_1 \in V} \dots \bigcup_{c_k \in V} \bigcup_{s \in \bar{V}} \bar{\text{O}}^1_{(c_1 \dots c_k s)}.$$

The Lebesgue measure of this set is equal to

$$\lambda(\bar{F}_{k+1} \cap \bar{\text{O}}^1_{[c_1 \dots c_k]}) = \frac{1}{\sigma_1 \sigma_2 \dots \sigma_k} \sum_{n=1}^{\infty} \frac{1}{(\sigma_k + u_n)(\sigma_k + u_n + 1)}$$

and for $\sigma_k > \sigma_0$ we have

$$\frac{\lambda(\bar{F}_{k+1} \cap \bar{\text{O}}^1_{[c_1 \dots c_k]})}{\left|\bar{\text{O}}^1_{[c_1 \dots c_m]}\right|} = (\sigma_k + 1)\varphi(\sigma_k) \geq \gamma \frac{\sigma_k + 1}{\sigma_k + u_1} \geq \frac{2\gamma}{1 + u_1}.$$

Then

$$\frac{\lambda(\bar{F}_{k+1})}{\lambda(F_k)} \geq c, \quad \text{where} \quad c = \frac{2\gamma}{1 + u_1}.$$

Hence, according to corollary 7, $\lambda(C[\bar{\text{O}}^1, V] = 0$. So, the lemma is proved. □

4. *For example, condition* (19) *is fulfilled when the elements of the set* V *form an arithmetical progression with difference* $d \geq 2$ *(see* [9]*).*

5. *If the set*

$$\bar{V} = \mathbb{N} \setminus V = \left\{b_1^1, b_2^1, b_2^2, b_3^1, b_3^2, b_3^3, \dots, b_n^1, b_n^2, \dots, b_n^n, \dots\right\}$$

has the properties

(1) $b_n^{i+1} - b_n^i = 1$,

(2) $b_{n+1}^1 - b_n^n = n + 1$, $n \in \mathbb{N}$,

then the set $C[\bar{\text{O}}^1, V]$ *is of zero Lebesgue measure.*

Proof. For any fixed cylindrical set $\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_k]}$ and corresponding σ_k we consider the sum

$$\begin{aligned}\varphi(\sigma_k) &= \sum_{c\in\bar{V}} \frac{1}{(\sigma_k+c)(\sigma_k+c+1)} = \sum_{n=1}^{\infty}\sum_{i=1}^{n} \frac{1}{(\sigma_k+b_n^i)(\sigma_k+b_n^i+1)}\\ &= \sum_{n=1}^{\infty}\left(\frac{1}{\sigma_k+b_n^1}-\frac{1}{\sigma_k+b_n^n+1}\right) = \sum_{n=1}^{\infty}\frac{b_n^n-b_n^1+1}{(\sigma_k+b_n^1)(\sigma_k+b_n^n+1)}\\ &= \sum_{n=1}^{\infty}\frac{n}{(\sigma_k+b_n^1)(\sigma_k+b_n^n+1)} \geq \sum_{n=1}^{\infty}\frac{n}{(\sigma_k+b_n^1)(\sigma_k+b_{n+1}^1)}\\ &= \sum_{n=1}^{\infty}\frac{n}{2n}\left(\frac{1}{\sigma_k+b_n^1}-\frac{1}{\sigma_k+b_{n+1}^1}\right) = \frac{1}{2(\sigma_k+b_1^1)}.\end{aligned}$$

Thus we have

$$\varphi(\sigma_k) \geq \frac{1}{2(\sigma_k+b_1^1)},$$

and the theorem follows from Lemma 7. □

6. *If the set*

$$\bar{V} = \mathbb{N}\setminus V = \left\{b_1^1, b_1^2, \ldots, b_1^{2^2}, b_2^1, b_2^2, \ldots, b_2^{2^3}, \ldots, b_n^1, b_n^2, \ldots, b_n^{2^{n+1}}, \ldots\right\}$$

has the properties

(1) $b_n^{i+1}-b_n^i = d$,

(2) $b_{n+1}^1 - b_n^{2^{n+1}} = d+1,\ n\in\mathbb{N}$,

then the set $C[\bar{\mathrm{O}}^1, V]$ *is of zero Lebesgue measure.*

Proof. For any fixed cylindrical set $\bar{\mathrm{O}}^1_{[c_1c_2\ldots c_k]}$ and corresponding σ_k we consider the sum

$$\varphi(\sigma_k) = \sum_{c\in\bar{V}} \frac{1}{(\sigma_k+c)(\sigma_k+c+1)} = \sum_{n=1}^{\infty}\sum_{i=1}^{2^{n+1}} \frac{1}{(\sigma_k+b_n^i)(\sigma_k+b_n^i+1)}.$$

We have the following estimate

$$\begin{aligned}\varphi(\sigma_k) &\geq \sum_{n=1}^{\infty}\Big(\sum_{i=1}^{2^{n+1}-1}\frac{1}{(\sigma_k+b_n^i)(\sigma_k+b_n^{i+1})} - \frac{1}{(\sigma_k+b_n^{2^{n+1}})(\sigma_k+b_n^{2^{n+1}}+d)}\Big)\\ &= \frac{1}{d}\sum_{n=1}^{\infty}\left(\frac{1}{\sigma_k+b_n^1}-\frac{1}{\sigma_k+b_n^{2^{n+1}}+d}\right)\\ &= \frac{1}{d}\sum_{n=1}^{\infty}\frac{b_n^{2^{n+1}}-b_n^1+d}{(\sigma_k+b_n^1)(\sigma_k+b_n^{2^{n+1}}+d)}\end{aligned}$$

$$= \frac{1}{d}\sum_{n=1}^{\infty}\frac{2^{n+1}d}{(\sigma_k+b_n^1)(\sigma_k+b_n^{2^{n+1}}+d)} \geq \frac{1}{d}\sum_{n=1}^{\infty}\frac{2^{n+1}d}{(\sigma_k+b_n^1)(\sigma_k+b_{n+1}^1)}$$

$$= \frac{1}{d}\sum_{n=1}^{\infty}\frac{2^{n+1}d}{2^{n+1}d+1}\left(\frac{1}{\sigma_k+b_n^1}-\frac{1}{\sigma_k+b_{n+1}^1}\right)$$

$$\geq \frac{1}{d}\sum_{n=1}^{\infty}\frac{2^{n+1}d}{2^{n+1}d+2^{n+1}d}\left(\frac{1}{\sigma_k+b_n^1}-\frac{1}{\sigma_k+b_{n+1}^1}\right) = \frac{1}{2d(\sigma_k+b_1^1)}.$$

Thus we have

$$\varphi(\sigma_k) \geq \frac{1}{2d(\sigma_k+b_1^1)},$$

and the theorem follows from Lemma 7. □

5. *If the spectrum of the ξ is determined by a set $\bar{V}$ which fulfills the conditions of theorems* 5 *or* 6*, then the probability distribution ξ is a singular distribution of Cantor type.*

7. *If sets $V = \{v : p_v = 0\}$ and $\bar{V} \equiv \mathbb{N}\backslash V = \{u : p_u \neq 0\}$ satisfy conditions of Theorems* 5 *or* 6*, then F is a singular function of Cantor type. Moreover, it is monotonic if $p_u > 0$ for any $u \in \bar{V}$ and non-monotonic if there exists $p_u < 0$.*

6. Fractal properties of the function F

For thinner analysis of essential sets for the function F we need notions of the fractal analysis (the theory of the Hausdorff measures of fractional orders and the theory of the metric Hausdorff–Besicovitch dimension [14, p. 53–56] or [4, p. 27–33]).

Let E be a bounded set from the space $\mathbb{R}^1$. The number

$$d(E) = \sup_{x,y\in E}|x-y|$$

is called the *diameter* of E. Let Φ be a family of subsets of the space $\mathbb{R}^1$ such that for any set $E \subset \mathbb{R}^1$ and for any $\varepsilon > 0$ there exists an at most countable ε-covering $\{E_j\}$ of the set E such that $E_j \in \Phi$ and $d(E_j) \leq \varepsilon$.

For any bounded set $E \subset \mathbb{R}^1$, for any $\alpha > 0$ and $\varepsilon > 0$ let

$$m_\varepsilon^\alpha(E,\Phi) = \inf_{d(E_j)\leq\varepsilon}\left\{\sum_j d^\alpha(E_j)\right\},$$

where the infimum is taken over all at most countable ε-coverings $\{E_j\}$ of the set E by sets $E_j \in \Phi$.

3. The non-negative number

$$H^\alpha(E,\Phi) = \lim_{\varepsilon\to 0} m_\varepsilon^\alpha(E,\Phi) = \sup_{\varepsilon>0} m_\varepsilon^\alpha(E,\Phi)$$

is called the *α-dimensional Hausdorff measure* (or *H^α-Hausdorff measure*) of the set E with respect to the family of coverings Φ.

8. *The Hausdorff measure has the following properties:*

(1) $H^\alpha\Big(\bigcup\limits_i E_i, \Phi\Big) \le \sum\limits_i H^\alpha(E_i, \Phi)$;

(2) If $\alpha_1 < \alpha_2$ *then* $H^{\alpha_1}(E, \Phi) \ge H^{\alpha_2}(E, \Phi)$;

(3) If $H^{\alpha_1}(E, \Phi) = 0$ *then* $H^{\alpha_2}(E, \Phi) = 0$ *for* $\alpha_1 < \alpha_2$;

(4) If $H^{\alpha_2}(E, \Phi) = \infty$ *then* $H^{\alpha_1}(E, \Phi) = \infty$ *for* $0 < \alpha_1 < \alpha_2$.

4. The non-negative number

$$\alpha_0(E, \Phi) = \sup\{\alpha : H^\alpha(E, \Phi) = +\infty\} = \inf\{\alpha : H^\alpha(E, \Phi) = 0\}$$

is called the *Hausdorff–Besicovitch dimension* of the set E with respect to the family of coverings Φ.

Let us give some properties of the Hausdorff–Besicovitch dimension:

(1) $\alpha_0(E, \Phi) = 0$ for any at most countable set E;

(2) $\alpha_0(E_1, \Phi) \le \alpha_0(E_2, \Phi)$ if $E_1 \subset E_2$;

(3) $\alpha_0(\bigcup\limits_n E_n, \Phi) = \sup\limits_n \alpha_0(E_n, \Phi)$;

(4) if E_1 and E_2 are affine equivalent (in particular, similar) sets then $\alpha_0(E_1, \Phi) = \alpha_0(E_2, \Phi)$.

Let Φ be a class of all intervals or closed intervals. Then the α-dimensional Hausdorff measure and the Hausdorff–Besicovitch dimension of set E are denoted by $H^\alpha(E)$ and $\alpha_0(E)$ respectively.

9. *If* $1 < m$ *is a fixed positive integer,* $V = \{1, 2, \ldots, m\}$, *then the set* $C \equiv C[\bar{\text{O}}^1, V]$ *is anomalously fractal, i.e.,*

$$\alpha_0(C) = 0.$$

Proof. It is evident that the set C is a subset of the set

$$F_n = \bigcup_{i_1=1}^{m} \ldots \bigcup_{i_n=1}^{m} \bar{\text{O}}^1_{[i_1 \ldots i_n]},$$

which is a union of m^n cylinders of the rank n, and $\bar{\text{O}}^1_{[\underbrace{11\ldots1}_n]}$ is a longest cylinder, namely:

$$\left|\bar{\text{O}}^1_{[\underbrace{11\ldots1}_n]}\right| = \frac{1}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \cdot (n+1)} = \frac{1}{(n+1)!} \equiv \varepsilon_n.$$

Then

$$m^\alpha_{\varepsilon_n}(C) \le \frac{m^n}{((n+1)!)^\alpha}$$

and

$$H^\alpha(C) \leq \lim_{n\to\infty} \frac{m^n}{((n+1)!)^\alpha} = \lim_{n\to\infty} \prod_{i=1}^{n} \frac{m}{(i+1)^\alpha} =$$

$$= \prod_{i=1}^{M} \frac{m}{(i+1)^\alpha} \prod_{i=M+1}^{\infty} \frac{m}{(i+1)^\alpha}.$$

Since for any m there exists M such that for all $i > M$ inequality

$$m < (i+1)^\alpha$$

holds, the last infinite product is equal to 0 for any fixed m and $\alpha \in (0,1]$. So, $H^\alpha(C) = 0$ for all $\alpha \in (0,1]$ and hence $\alpha_0(C) = 0$. □

8. *If $1 < m$ is a fixed positive integer, $V = \{1, 2, \ldots, m\}$ $V_0 \subset V$, then the set $C[\bar{\text{O}}^1, V_0]$ is anomalously fractal.*

In general,

$$H^\alpha(E, \Phi_1) \neq H^\alpha(E, \Phi_2) \quad \text{as well as} \quad \alpha_0(E, \Phi_1) \neq \alpha_0(E, \Phi_2).$$

If $\Phi_1 \subset \Phi_2$ (i.e., any set from the family Φ_1 belongs to the family Φ_2) then it is evident that

$$H^\alpha(E, \Phi_1) \geq H^\alpha(E, \Phi_2) \quad \text{and} \quad \alpha_0(E, \Phi_1) \geq \alpha_0(E, \Phi_2).$$

However, if Φ_1 is the family of all subsets, Φ_2 is the family of all open subsets and Φ_3 is a family of all closed subsets of $\mathbb{R}^1$ then

$$H^\alpha(E, \Phi_1) = H^\alpha(E, \Phi_2) = H^\alpha(E, \Phi_3)$$

and

$$\alpha_0(E, \Phi_1) = \alpha_0(E, \Phi_2) = \alpha_0(E, \Phi_3)$$

for any set $E \subset \mathbb{R}^1$.

One of the traditional problems of the theory of the Hausdorff–Besicovitch dimension is a question if class of sets Φ is enough for equality

$$\alpha_0(E, \Phi) = \alpha_0(E).$$

One can prove that the set $\mathfrak{U}$ of all cylindrical sets corresponding to $\bar{\text{O}}^1$-representation is not enough, i.e., it is easy to construct set E such that $\alpha_0(E, \mathfrak{U}) = \alpha_0(E)$.

Let $\mathfrak{W}$ be a class of all connected sets such that they are unions of cylinders of the same rank $m+1$ belonging to the same cylinder of the rank m. That is the class $\mathfrak{W}$ consists of sets of the form:

(1) $\bar{O}^1_{[c_1c_2\dots c_m]}$, (2) $\bigcup\limits_{i=n}^{\infty}\bar{O}^1_{[c_1c_2\dots c_m i]}$,

(3) $\bigcup\limits_{i=1}^{n}\bar{O}^1_{[c_1c_2\dots c_m i]}$, (4) $\bigcup\limits_{i=k}^{n}\bar{O}^1_{[c_1c_2\dots c_m i]}$

for all positive integer k, m, n and sequences $(c_1, c_2, \dots, c_m)$ of positive integers.

It is clear that for $n = 1$ set (2) is a set (1), and for $k = 1$ set (4) is a set (3).

Let $\mathfrak{W}_\varepsilon$ be a class of sets belonging to $\mathfrak{W}$ such that their lengths does not exceed ε.

10. *For any interval $u \subset (0, 1]$ there exists at most four sets belonging to $\mathfrak{W}_{|u|}$ and covering u.*

Proof. Let $u = (a, b)$. The points a and b may belong

(1) to different cylindrical sets of rank 1,

(2) to the same cylinder of rank 1.

Consider case (1). Let $a \in \bar{O}^1_{[a_1]}$, $b \in \bar{O}^1_{[b_1]}$, $c = \sup \bar{O}^1_{[a_1]}$, $d = \inf \bar{O}^1_{[b_1]}$. Since $a < b$, we have $a_1 > b_1$ and $a < d < b$.

Two cases are possible: $a_1 - b_1 > 1$ $a_1 - b_1 = 1$.

1.1. Let $a_1 - b_1 > 1$. Then $(a, b) = (a, d] \cup (d, b)$.

If $a = \inf \bar{O}^1_{[a_1]}$, then $[a, d] = \bigcup\limits_{i=b_1+1}^{a_1} \bar{O}^1_{[i]} \in \mathfrak{W}_{d-a} \subset \mathfrak{W}_{b-a}$.

If $a \in \bar{O}^1_{(a_1)}$, then (a, d) is covered by two sets from $\mathfrak{W}_{d-a}$, namely:

$$\bar{O}^1_{[a_1]} \quad \text{and} \quad \bigcup_{j=b_1+1}^{a_1-1} \bar{O}^1_{[j]}.$$

So, to cover $[a, d]$ is enough two sets from $\mathfrak{W}_{d-a}$, hence, two sets from $\mathfrak{W}_{b-a}$.

Consider $[d, b]$. If $b = \sup \bar{O}^1_{[b_1]}$, then $[d, b] = \bar{O}^1_{[b_1]} \in \mathfrak{W}_{b-d} \subset \mathfrak{W}_{b-a}$.

If $b \in \bar{O}^1_{(b_1)}$, then we consider cylinders of rank 2 $\bar{O}^1_{[b_1 j]}$ belonging to $\bar{O}^1_{[b_1]}$.

If $b = \sup \bar{O}^1_{[b_1 n]}$, then $[d, b] = \bigcup\limits_{j=1}^{n} \bar{O}^1_{[b_1 j]} \in \mathfrak{W}_{b-d}$.

If $b \in \bar{O}^1_{(b_1 n)}$, then $[d, b]$ is covered by: a) two sets from $\mathfrak{W}_{b-a}$:

$$\bigcup_{j=1}^{n-1} \bar{O}^1_{[b_1 j]} \quad \text{and} \quad \bar{O}^1_{[b_1 n]}, \quad \text{if } n > 1,$$

b) one set $\bar{O}^1_{[b_1 1]}$, if $n = 1$, since $\left|\bar{O}^1_{[b_1 1]}\right| < \left|\bar{O}^1_{[b_1+1]}\right|$, $\bar{O}^1_{[b_1+1]} \subset [a, b]$.

Hence, it is enough two sets from $\mathfrak{W}_{b-a}$ for covering $[d, b]$ and at most four sets for covering $[a, b]$.

1.2. Let $a_1 - b_1 = 1$. Then $c = d$, $a \in \bar{O}^1_{(a_1)}$.

Consider $[a, d]$. If $a = \inf \bar{O}^1_{[a_1 k]}$, then

$$[a, d] = \bigcup_{j=k}^{\infty} \bar{O}^1_{[a_1 j]} \in \mathfrak{W}_{d-a} \subset \mathfrak{W}_{b-a}.$$

If $a \in \bar{O}^1_{(a_1 k)}$, then $[a, d]$ is covered by two sets from $\mathfrak{W}_{b-a}$, namely:

$$\bar{O}^1_{[a_1 k]} \quad \text{and} \quad \bigcup_{j=k+1}^{\infty} \bar{O}^1_{[a_1 j]},$$

since according to equality (11) we have $\left|\bar{O}^1_{[a_1 k]}\right| < \sum_{j=k+1}^{\infty} \left|\bar{O}^1_{[a_1 j]}\right|$.

Hence, it is enough two sets from $\mathfrak{W}_{b-a}$ for covering $[a, d]$.

Now let us consider $[d, b]$. If $b = \sup \bar{O}^1_{[b_1 n]}$, then

$$[d, b] = \bigcup_{j=1}^{n} \bar{O}^1_{[b_1 j]} \in \mathfrak{W}_{b-d}.$$

If $b \in \bar{O}^1_{(b_1 n)}$ $n > 1$, then $[d, b]$ is covered by two sets from $\mathfrak{W}_{b-a}$, namely:

$$\bigcup_{j=1}^{n-1} \bar{O}^1_{[b_1 j]} \quad \text{and} \quad \bar{O}^1_{[b_1 n]},$$

since $\left|\bar{O}^1_{[b_1 n]}\right| < \left|\bar{O}^1_{[b_1 1]}\right|$, $\bar{O}^1_{[b_1 1]} \subset [d, b]$.

If $b \in \bar{O}^1_{(b_1 1)}$, then we consider cylinders $\bar{O}^1_{[b_1 1 j]}$ of rank 3 belonging to $\bar{O}^1_{[b_1 1]}$. In this case $[d, b]$ is covered by at most two sets from $\mathfrak{W}_{b-a}$, namely: a) one set:

$$\bigcup_{j=s}^{\infty} \bar{O}^1_{[b_1 1 j]},$$

if $b = \sup \bar{O}^1_{[b_1 1 s]}$ b) two sets

$$\bigcup_{j=s+1}^{\infty} \bar{O}^1_{[b_1 1 j]} \quad \text{and} \quad \bar{O}^1_{[b_1 1 s]},$$

if $b \in \bar{O}^1_{(b_1 1 s)}$, because the length of last set is less than the diameter of the first set. Hence, it is enough two sets from $\mathfrak{W}_{b-a}$ for covering $[d, b]$ and four sets for covering of whole $[a, b]$.

2. If a and b belong to the same cylinder of rank 1, then there exists cylinder $\bar{O}^1_{[c_1 c_2 \ldots c_m]}$ of some rank m containing numbers a and b, but there does not exist a cylinder of rank $m + 1$ containing these numbers.

If m is even, then to prove the lemma it is enough to repeat the procedure used in the case 1 with the cylinder $\bar{O}^1_{[c_1 c_2 \ldots c_m]}$ instead of $[0, 1]$.

If m is odd number, one can proceed analogous. Here the numbers a and b, c and d are interchanged. $\square$

8. *The class of sets $\mathfrak{W}$ is enough for the determination of the Hausdorff–Besicovitch dimension of any Borel set $E \subset [0,1]$, i.e.,*

$$\alpha_0(E, \mathfrak{W}) = \alpha_0(E). \tag{20}$$

Proof. From lemma 10 it follows

$$m_\varepsilon^\alpha(E, \mathfrak{W}) \leq 4 m_\varepsilon^\alpha(E).$$

Indeed, for any closed interval u belonging to the covering E, there exists at most four sets ω_1, ω_2, ω_3, ω_4 $\mathfrak{W}$ such that

$$|\omega_i|^\alpha \leq |u|^\alpha \quad \text{for any } \alpha \in (0,1).$$

On the other hand,

$$m_\varepsilon^\alpha(E) \leq m_\varepsilon^\alpha(E, \mathfrak{W}),$$

because in determination of $m_\varepsilon^\alpha(E)$ the infimum is taken on wider class of coverings containing also sets from $\mathfrak{W}$. Thus,

$$m_\varepsilon^\alpha(E) \leq m_\varepsilon^\alpha(E, \mathfrak{W}) \leq 4 m_\varepsilon^\alpha(E)$$

for any $\varepsilon > 0$. Hence,

$$H^\alpha(E) \leq H^\alpha(E, \mathfrak{W}) \leq 4 H^\alpha(E),$$

i.e., $H^\alpha(E)$ and $H^\alpha(E, \mathfrak{W})$ simultaneously over α take the values 0 and ∞. It means that equality (20) holds. This proves the theorem. □

We mean *fractal properties* of function as:

(1) fine metric properties of essential for function sets related to the Hausdorff–Besicovitch dimension (the spectrum of singular probability distribution function, level sets of nowhere monotonic function et al.),
(2) fractal properties of graph of function as a set of space $\mathbb{R}^2$,
(3) transformation of dimension of sets if function is strictly monotonic.

7. Function F as transformation of $[0,1]$

9. *If $p_n > 0$ for all $n \in \mathbb{N}$ then the function F is a transformation of $[0,1]$ (i.e., bijective mapping of $[0,1]$ onto oneself) such that for any sequence (p_n) it does not preserve the Hausdorff–Besicovitch dimension, i.e., there exists $E \subset [0,1]$ such that*

$$\alpha_0(E) \neq \alpha_0(F(E)).$$

Proof. Let us consider the set

$$C[\bar{\text{O}}^1, V] = \{x : x = \bar{\text{O}}^1(g_1(x), \ldots, g_n(x), \ldots),$$
$$g_n(x) \in V = \{m+1, m+2, \ldots\} \equiv N_m^\infty\}$$

where $1 < m$ is a fixed positive integer.

It is proved in [16] that the Lebesgue measure of the set $C[\bar{\text{O}}^1, V]$ is positive, hence

$$\alpha_0(C[\bar{\text{O}}^1, V]) = 1.$$

On the other hand, the Hausdorff–Besicovitch dimension of the set $F(E)$ is a solution of the equation

$$\sum_{i=m+1}^{\infty} p_i^x = 1.$$

It is easy to see that this solution is less than 1. □

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